

# THE BUNDLE OF A LATTICE GAUGE FIELD

CLAUDIO MENESES AND JOSÉ A. ZAPATA

**ABSTRACT.** Given a smooth manifold  $M$  and a Lie group  $G$ , we consider parallel transport maps –groupoid homomorphisms from a path groupoid in  $M$  to  $G$ – as an alternative description of principal  $G$ -bundles with smooth connections on them. Using a cellular decomposition  $\mathcal{C}$  of  $M$ , and a system of paths associated to  $\mathcal{C}$ , we define a homotopical equivalence relation of parallel transport maps, leading to the concept of an extended lattice gauge (ELG) field. A lattice gauge field, as used in Lattice Gauge Theory, is part of the data contained in an ELG field, but the latter contains additional topological information of local nature, sufficient to reconstruct a principal  $G$ -bundle up to equivalence, in the spirit of Barrett [3]. Our constructed space of ELG fields on a given pair  $(M, \mathcal{C})$  is a covering space for a finite dimensional Lie group, whose connected components parametrize equivalence classes of principal  $G$ -bundles on  $M$ . Following a theorem of Pachner [16], we give a criterion to determine when two ELG fields over different cell decompositions  $\mathcal{C}$  and  $\mathcal{C}'$  define equivalent bundles. As a first concrete physical application, we define a simple operation that captures the geometric essence of the 't Hooft loop operator in the theory of quantum gauge fields, and study its relation with the bundle structure of an ELG field.

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## 1. INTRODUCTION

The main objective of this paper is to study how the topology of principal bundles may be incorporated into the formulation of Lattice Gauge Theory. The standard geometric approach to the study of field theories with gauge symmetry, such as the Yang-Mills theory, considers a principal  $G$ -bundle  $P \rightarrow M$  over a smooth manifold  $M$ , from which a space of “states”  $\mathcal{A}_P/\mathcal{G}_P$ , consisting of smooth connections in  $P$ , modulo the action of the group of gauge transformations, is induced. However, in the standard formulation of Lattice Gauge Theory on  $\mathbb{R}^n$  (or more generally,  $\mathbb{R}^n/\Lambda$  for a lattice  $\Lambda$ ) [5], the discrete analog of a gauge field, the so-called *lattice gauge field*, corresponds geometrically to the evaluation of the parallel transport of a connection (a gauge-invariant notion) over a discrete collection of paths, generated by a lattice. Our motivation is the observation that, while such notion captures the essence of a discrete gauge field (a question of a local nature), it is insufficient to characterize the topology of  $P$ : the information on how to glue together local trivializations is missing. While the question of construction of topological charges in special cases and in reduced regimes is known (e.g. [15], where a physical application is given in the case  $\dim M = 2$ ,  $G = \mathrm{SO}(2)$ , and [17], where the focus is in the non-abelian case  $\dim M = 4$ ,  $G = \mathrm{SU}(2)$ ), it is our impression that the previous problem has not yet been explored comprehensively in the existent literature.

In the continuum, the Barrett-Kobayashi construction [3] ensures that a suitable axiomatization of the notion of *holonomy* of a connection [9, 1], which is based on the structure of the space of based loops in  $M$  modulo thin equivalence, is sufficient to reconstruct a pair  $(P, A)$ , consisting of a principal  $G$ -bundle and a smooth connection on it. Since a lattice gauge field may be conceived as a discretization of a smooth holonomy map, It is then natural to infer, from the very nature of the Barrett-Kobayashi construction, that a refined notion of discrete gauge field, which we call an *extended lattice gauge field*, can be given instead, as an equivalence class of gauge fields, under a suitable topological equivalence relation, in such a way that the topology of a principal  $G$ -bundle prevails through the decimation process. Such is the basic idea that we exploit.

The general problem of classification of isomorphism classes of principal  $G$ -bundles over  $M$  has a natural homotopical interpretation in terms of the classifying space  $BG$  [12, 14], and it is ultimately related to the structure of the homotopy groups  $\pi_1(G, e), \dots, \pi_{n-1}(G, e)$ . An alternative description of the classification problem is given by the Čech cohomology with coefficients in  $G$ , and moreover, for the  $n$ -sphere,  $n \geq 2$ , the answer to the problem is greatly simplified, since there is a correspondence between equivalence classes of smooth principal  $G$ -bundles  $P \rightarrow S^n$ , and isotopy classes of smooth functions  $g : S^{n-1} \rightarrow G$ , the *clutching* maps for  $P$ . Based on those facts, we provide an alternative hybrid homotopical interpretation of an equivalence class of smooth principal  $G$ -bundles, which is similar in spirit

to the general considerations of Segal in [20], and has the advantage of being constructive, more explicit, and more amenable for computations than the standard one. The essence of the problem is translated into a combinatorial one. The starting point consists on choosing an auxiliary “scaffolding” structure on  $M$ , namely a special but generic type of cellular decomposition  $\mathcal{C}$ , dual to a triangulation in  $M$ , together with a network of paths  $\Gamma$  joining based points for every pair of neighboring cells. Then we adapt the notion of an equivalence class of Čech cocycles, to a homotopy class of collections of compatible clutching maps. The intrinsic triangulation serves the practical purpose of organizing the new homotopy data, as the clutching maps considered are defined over the  $(n - 1)$ -skeleton of  $\mathcal{C}$ , and the compatibility conditions they satisfy are systematically encoded over the  $(n - 2)$ -skeleton of  $\mathcal{C}$ . Most importantly, the resulting structure is tailored to fit into our subsequent formalism for the discretization of gauge fields.

Our first result, theorem 1, is a reconstruction theorem, in the spirit of Barrett [3], adapted to the notion of parallel transport of local path families. Equipped with it, we introduce the notion of an *extended lattice gauge field*. Then, we proceed to dissect the extended lattice fields, into elementary local pieces (theorem 2). Such pieces consist of the standard LGT data, namely, the values of parallel transport over a network of paths in  $M$ , but also contain homotopy classes of extensions of glueing maps from the boundary of a  $(k + 1)$ -cell –a  $k$ -sphere– to its interior. In the process, we are able to identify the minimal local topological data, contained in an extended lattice gauge field, that is sufficient to reconstruct a given bundle, which we name the *core* of an extended lattice gauge field (corollary 1). Such a glossary of local data turns out to reveal a *covering space* structure in the space of extended lattice gauge fields for a given triple  $(M, \mathcal{C}, G)$ , fibering over the space of standard LGT data –a Lie group– (corollary 2). The group of deck transformations of such a covering space is modeled on certain subgroups of an inductive product of homotopy groups, acting on the previously mentioned homotopy classes of extensions of glueing maps. Furthermore, the connected components of the covering space are in bijective correspondence with the equivalence classes of principal  $G$ -bundles on  $M$ . As an interesting by-product, we are able to reconstruct the space  $\check{H}^1(M, \underline{G})$  of equivalence classes of principal  $G$ -bundles on  $M$  as a *homogeneous space* for such group of deck transformations (corollary 3). While not the main objective of our present investigation, the characterization of the space of isomorphism classes of principal  $G$ -bundles as a homogeneous space, which is inspired by the holonomy point of view, provides a new promising structure, of combinatorial nature, to organize and classify its internal structure. In particular, it should be possible to reconstruct the Chern-Weil theory of characteristic classes from such a combinatorial point of view. Such questions deserve to be investigated separately, and we are planning to address them in the near future.

We have described results which build on the introduction of an additional structure over a manifold  $M$ , namely a triangle-dual cellular decomposition. Our results concerning principal bundles over  $M$  can be freed from such an auxiliary structure. We use a dual version of the fundamental theorem of U. Pachner [16], relating any two triangulations of  $M$  by a series of elementary transformations, to provide a criterion that determines when two extended lattice gauge fields, defined over different cellular decompositions, define equivalent principal  $G$ -bundles (theorem 3). It is in that way that our results are freed from their previous dependence on a cellular decomposition.

We complement the previous results with a concrete physical application. We define a simple geometric operation on a gauge field, that captures the geometric essence of the 't Hooft loop operator in the theory of quantum gauge fields. In theorem 4, We study its effect in the bundle structure of the extended lattice gauge fields. Interestingly, our approach leads to the notion of a non-abelian gerbe, as the natural geometric framework for the study the 't Hooft operation.

The work is organized as follows. In section 2, we introduce the special cellular decompositions we will work with. Section 3 is a short introduction to path structures and local path families, which are exploited in section 4, in order to introduce the cellular parallel transport maps, the reconstruction theorem for bundles with connection, and the extended lattice gauge fields. Section 5 constitutes the heart of the article, where the relation between extended lattice gauge fields and isomorphism classes of principal bundles is studied, and is complemented with section 6, describing explicitly the extended lattice gauge fields in small dimensions. Section 7 describes the dependence of the bundle structure of the extended lattice gauge fields on different cellular decompositions, while section 8 concludes with a study of the 't Hooft loop operation in the present topological context. In addition, two appendices complement our work. Appendix A is a brief introduction to cellular decompositions, where the genericity of the triangle-dual ones is justified, while appendix B contains a proof of the correspondence between the space of equivalence classes of principal  $G$ -bundles, and the space of equivalence classes of local homotopy data in the core of an ELG field.

## 2. TRIANGLE-DUAL CELLULAR DECOMPOSITIONS

Let  $M$  be an  $n$ -dimensional smooth manifold. It is a classical result of J. H. C. Whitehead that  $M$  admits a piecewise-linear (P.L.) structure, namely, a triangulation  $\Delta$  for which the link of any simplex is a piecewise-linear sphere, and moreover, any two such P.L. structures are related by a piecewise-smooth bijection. It would be sufficient to assume the piecewise-smoothness of any such  $\Delta$ . For any triangulation  $\Delta$ , we will denote its 0-simplices by  $v, w, \dots$  and unless otherwise stated, a given 1-simplex by  $\tau$ . The letter  $\sigma$  will denote an arbitrary  $k$ -simplex. We will assume henceforth that  $M$  is orientable, and that a choice of orientation has been given to it.

**Definition 1.** A cellular decomposition  $\mathcal{C} = \sqcup_{k=0}^n \mathcal{C}_k$  of  $M$  is called *triangle-dual* if there exist an open cover  $\mathfrak{U} = \{\mathcal{U}_v\}_{c_v \in \mathcal{C}_n}$  of  $M$  with the following properties:

- (i) For every  $\sigma \in N(\mathfrak{U})$  (the nerve of  $\mathfrak{U}$ , an abstract simplicial complex), the open set  $\mathcal{U}_\sigma = \cap_{v \in \sigma} \mathcal{U}_v$  is contractible,
- (ii) The geometric realization of  $N(\mathfrak{U})$  is a P.L. structure for  $M$ . In particular,  $N(\mathfrak{U})$  is pure and  $\cap_{i=0}^{n+1} \mathcal{U}_{v_i} = \emptyset$  for all pairwise-different  $c_{v_0}, \dots, c_{v_{n+1}} \in \mathcal{C}_n$ ,
- (iii) There is a 1-to-1 correspondence between the  $k$ -simplices  $\sigma \in N(\mathfrak{U})$  and the  $(n-k)$ -cells  $c_\sigma \in \mathcal{C}_{n-k}$ , in such a way that  $\overline{c_\sigma} \subset \mathcal{U}_\sigma$ .

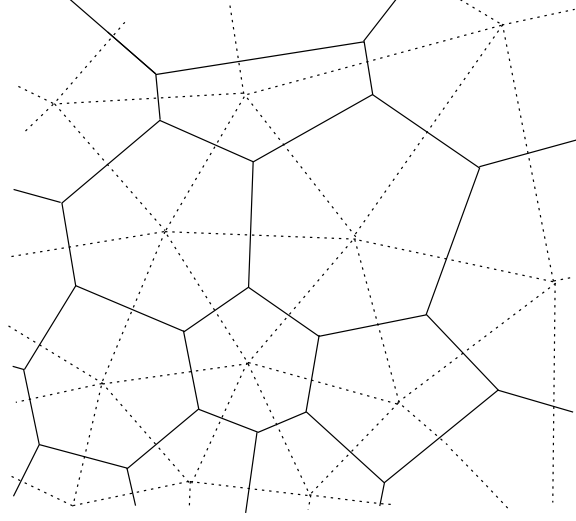
*Remark 1.* The motivation behind this definition is that, for any choice of P.L. structure  $\Delta$  of  $M$ , the dual cellular decomposition  $\Delta^\vee$  of  $M$ , which is constructed in terms of the canonical barycentric subdivision of  $\Delta$ , is triangle-dual with respect to the open cover of  $M$  given by the stars of the vertices of  $\Delta$  (figure 1). Conversely, given a triangle-dual cellular decomposition  $\mathcal{C}$  of  $M$ , it is also possible to provide a full characterization of a P.L. structure  $\Delta$  such that  $\Delta^\vee = \mathcal{C}$  if we are allowed to strengthen the choice of the cover  $\mathfrak{U}$ . Namely, by a *star-like cover* we mean an open cover where, moreover,  $\mathcal{D}_k = \mathcal{S}_k - \mathcal{S}_{k-1}$ ,  $k = 0, \dots, n$  is a disjoint union of embedded smooth  $k$ -disks, for  $\mathcal{S}_{-1} = \emptyset$ , and

$$\mathcal{S}_k = M \setminus \bigcup_{|\sigma|=k+2} \mathcal{U}_\sigma, \quad k = 0, \dots, n.$$

This way, the components of each  $\mathcal{D}_k$  correspond to the interiors of the  $k$ -simplices of  $\Delta$ , and  $\mathcal{S}_k$  corresponds to the  $k$ -th skeleton of a triangulation  $\Delta$ . Observe that, in particular, for any  $k$ -cell  $c_\sigma \in \mathcal{C}_k$ , there is an induced triangle-dual cellular decomposition of  $\partial \overline{c_k}$  (a piecewise-smooth  $(k-1)$ -sphere in  $M$ ).

To complement the previous picture, and justify the naturality of triangle-dual cellular decompositions, a genericity property satisfied by them is stated and proved in lemma 7 in appendix A.

*Remark 2.* A very important combinatorial feature that distinguishes the triangle-dual cellular decompositions can be described as follows. When a cellular decomposition  $\mathcal{C}$  is triangle-dual, and  $c_\sigma \in \mathcal{C}_k$ , there are exactly  $n-k+1$   $(k+1)$ -cells  $c_{\sigma_1}, \dots, c_{\sigma_{n-k+1}} \in \mathcal{C}_{k+1}$  such that  $\cap_{j=1}^{n-k+1} \overline{c_{\sigma_j}} = \overline{c_\sigma}$ . This is so since, over the dual abstract simplicial complex  $\Delta$ , the  $n-k+1$  different  $(n-k-1)$ -simplices  $\sigma_1, \dots, \sigma_{n-k+1}$  determine the boundary  $\delta(\sigma)$  of a unique  $(n-k)$ -simplex  $\sigma$ . If necessary, one could denote such  $c_\sigma$  by  $c_{\sigma_1 \dots \sigma_{n-k+1}}$ . Following the same logic, we conclude that for every  $c_\sigma \in \mathcal{C}_k$ ,  $\overline{c_\sigma}$  can be expressed as the intersection of exactly  $n-k+1$  closures of  $n$ -cells  $\overline{c_{v_1}}, \dots, \overline{c_{v_{n-k+1}}}$ . As a consequence, one has the following. Whenever there is a triple  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\} \subset \mathcal{C}_{n-1}$  whose closures intersect nontrivially in  $\overline{c_\sigma}$ ,

FIGURE 1. A triangle-dual cellular decomposition  $\mathcal{C}$  on a surface.

for some  $c_\sigma \in \mathcal{C}_{n-2}$ , there is a unique triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathcal{C}_n$  such that

$$(2.1) \quad \overline{c_{\tau_1}} = \overline{c_{v_2}} \cap \overline{c_{v_3}}, \quad \overline{c_{\tau_2}} = \overline{c_{v_3}} \cap \overline{c_{v_1}}, \quad \overline{c_{\tau_3}} = \overline{c_{v_1}} \cap \overline{c_{v_2}},$$

and

$$(2.2) \quad \overline{c_{\tau_1}} \cap \overline{c_{\tau_2}} \cap \overline{c_{\tau_3}} = \overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} = \overline{c_\sigma}.$$

*Remark 3.* Over a triangle-dual cellular decomposition, an orientation in  $M$  induces an orientation on the elements of each complete descending flag in  $\mathcal{C}$ . Once a  $k$ -cell  $c_\tau$  has been given an orientation, there is an induced orientation on every  $(k-1)$ -cell  $c_\sigma \subset \partial \overline{c_\tau} \cong S^{k-1}$ . When the cellular decomposition is triangle-dual, the two different orientations that any  $k$ -cell  $c_\sigma$  may acquire can be described as follows. Every gapless descending flag ending in  $\overline{c_\sigma}$  is of the form

$$\overline{c_{v_1}} \supset \overline{c_{v_1}} \cap \overline{c_{v_2}} \supset \cdots \supset \bigcap_{j=1}^{n-k+1} c_{v_j},$$

and moreover, any other gapless flag would correspond to a permutation of the ordered  $(n-k+1)$ -tuple  $(c_{v_1}, \dots, c_{v_{n-k+1}})$ . Then, two different gapless flags ending at the same element induce the same orientation if and only if they differ by an even permutation.<sup>1</sup> Moreover, for every ordered triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\}$  as in remark 2, we can decree an orientation in the induced triple  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$  from the choice of flags  $c_{v_1} \supset \overline{c_{\tau_2}}$ ,  $c_{v_2} \supset \overline{c_{\tau_3}}$  and  $c_{v_3} \supset \overline{c_{\tau_1}}$ ,

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<sup>1</sup>The previous definition is compatible with the corresponding notion of orientation of the simplex  $\sigma$ .

in such a way that the orientations furtherly induced in  $c_\sigma$  coincide. We will say that the triple

$$(2.3) \quad \{\overline{c_{v_1}} \supset \overline{c_{\tau_2}} \supset \overline{c_\sigma}, \quad \overline{c_{v_2}} \supset \overline{c_{\tau_3}} \supset \overline{c_\sigma}, \quad \overline{c_{v_3}} \supset \overline{c_{\tau_1}} \supset \overline{c_\sigma}\}$$

has been given a *cyclic orientation*, which is clearly invariant under cyclic permutations of the indices. Given any  $c_\sigma \in \mathcal{C}_{n-2}$ , there are two choices of cyclic orientations in its corresponding triple of flags (2.3). Equivalently, one may reconstruct a cyclic orientation from a choice of orientation of the 2-simplex  $\sigma$ .

In the case when the manifold  $M$  is an Euclidean space or a quotient of it by a lattice (i.e., a cylinder or a torus), it would be of paramount importance to relate their standard cell decompositions generated by a square lattice in  $\mathbb{R}^n$  to the triangle-dual cell decompositions. Hence, we include the following lemma, justifying that the constructions and results of this work are also relevant and applicable in those standard cases.

**Lemma 1.** *The cell decomposition  $\mathcal{C}^0$  of  $\mathbb{R}^n$  induced by the rectangular lattice  $\Lambda^0 = \mathbb{Z}^n$  can be regarded as a degeneration of a family of triangle-dual cellular decompositions  $\mathcal{C}^\epsilon$ ,  $\epsilon \in (0, 1)$ .*

*Proof.* Setting a 1-1 correspondence between  $n$ -cells  $c_v^0$  in  $\mathcal{C}^0$  and the 0-subcell  $(i_1^v, \dots, i_n^v)$  in their closure, where for each  $k = 1, \dots, n$ ,

$$i_k^v = \min\{i_k : (i_1, \dots, i_n) \in \overline{c_v^0}\},$$

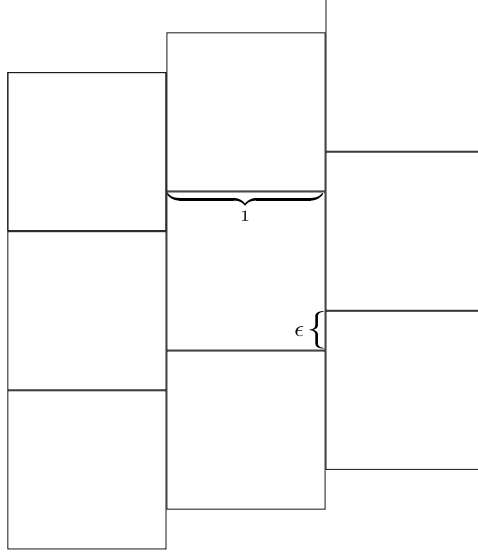
we can identify the  $n$ -cells in  $\mathcal{C}^0$  with the elements in  $\Lambda^0$ . For any  $\epsilon \in (0, 1)$ , consider the lattice  $\Lambda^\epsilon$  in  $\mathbb{R}^n$ , of  $n$ -tuples of real numbers of the form

$$(i_1, i_2 + \epsilon i_1, \dots, i_{n-1} + \epsilon i_{n-2}, i_n + \epsilon i_{n-1}), \quad i_1, \dots, i_n \in \mathbb{Z}.$$

Then, in particular,  $\Lambda^0 = \mathbb{Z}^n$ . Just as the lattice  $\Lambda^0$  acts as the group of translation symmetries of  $\mathcal{C}^0$ , the lattice  $\Lambda^\epsilon$  will act on  $\mathcal{C}^\epsilon$  as its symmetry group of translations. The  $n$ -cell  $c_v^\epsilon$ , as well as its closure, is then defined by translating  $c_v^0$  by the vector

$$\epsilon (0, i_1^v, \dots, i_{n-1}^v),$$

(figure 2). It should be remarked that although there is a 1-1 correspondence between  $n$ -cells in  $\mathcal{C}^0$  and  $\mathcal{C}^\epsilon$  by construction, new  $k$ -cells,  $k = 0, \dots, n-1$  are created in  $\mathcal{C}^\epsilon$  by subdivision of the boundary cells of each  $n$ -cell  $c_v^\epsilon$ . It is not difficult to see that, indeed,  $\mathcal{C}^\epsilon$  is triangle-dual for every  $\epsilon \in (0, 1)$ . Notice that over  $\mathcal{C}^0$ , each 0-cell is the common intersection of  $2n$  1-cells. The  $\epsilon$ -shifts introduced to define  $\mathcal{C}^\epsilon$  separate  $n-1$  of these 1-cells, leaving exactly  $n+1$  1-cells on each old and new 0-cell. The argument with the higher dimensional cells is similar, giving the combinatorial properties that determine a triangle-dual cell decomposition.  $\square$

FIGURE 2. The cell decomposition  $\mathcal{C}^\epsilon$  in  $\mathbb{R}^2$ .

### 3. CELLULAR PATH STRUCTURES

Given a piecewise-smooth path  $\gamma : [0, 1] \rightarrow M$ , we follow the convention of denoting  $\gamma(0) = s(\gamma)$  and  $\gamma(1) = t(\gamma)$  (the *source* and *target* of  $\gamma$ ).

**Definition 2.** We say that two piecewise-smooth paths  $\gamma$  and  $\gamma'$  in  $M$ , with  $s(\gamma) = s(\gamma')$  and  $t(\gamma) = t(\gamma')$ , are *thinly homotopic*, if the loop  $\gamma^{-1} \cdot \gamma'$  is thin, i.e. if there is a deformation retraction of  $\gamma^{-1} \cdot \gamma'$  to  $s(\gamma')$  for which the pullback of any 2-form in  $M$  to  $[0, 1] \times [0, 1]$  is zero [3]. A homotopy of paths consisting of thinly equivalent paths is called a *retracing*.<sup>2</sup>

Naturally, for every thin homotopy class  $[\gamma]$ , there is a unique inverse class  $[\gamma]^{-1}$  under path multiplication, in the sense for any representatives  $\gamma \cdot \gamma^{-1}$  and  $\gamma^{-1} \cdot \gamma$  are thinly null homotopic, namely, the thin homotopy class of the paths  $\gamma^{-1}(t) := \gamma(1-t)$ . When defined, the multiplication of thin homotopy classes of paths is associative, thus giving rise to the structure of a groupoid.

Let us fix, for every  $c_\sigma \in \mathcal{C}$ , a point  $p_\sigma \in c_\sigma$ , which will be referred as the base point of such cell. It readily follows from the previous definitions that for every  $k$ -simplex  $\sigma \in N(\mathfrak{U})$ ,  $\mathcal{U}_\sigma \setminus \{p_\sigma\}$  deformation retracts to  $\partial \overline{c_\sigma}$ , which is a piecewise-smooth  $(n - k - 1)$ -sphere. We will denote the collection of all base points in a given  $\overline{c_\sigma}$  by  $\mathcal{B}_\sigma$ , and the collection of all base points in  $\mathcal{C}$  by  $\mathcal{B}_\mathcal{C}$ .

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<sup>2</sup>A reparametrization of a path  $\gamma$  is a particular kind of retracing, but the latter is considerably more general. In particular, the equivalence of piecewise-smooth paths under retracing implies that a smooth representative for any  $[\gamma]$  can always be found.



**Definition 3.** For every  $c_\sigma \in \mathcal{C}$ , let  $\mathcal{P}_\sigma$  be the path groupoid of piecewise-smooth paths  $\gamma : [0, 1] \rightarrow \overline{c_\sigma}$ , such that  $s(\gamma), t(\gamma) \in \mathcal{B}_\sigma$ , modulo retracing, and let  $\mathcal{P}_\mathcal{C}$  be the path groupoid of piecewise-smooth paths  $\gamma : [0, 1] \rightarrow M$  such that  $s(\gamma), t(\gamma) \in \mathcal{B}_\mathcal{C}$ , modulo retracing. In particular, each of the groupoids  $\mathcal{P}_\sigma$  contains the respective loop groups  $\mathcal{L}(\overline{c_\sigma}, p_{\sigma'}) = \Omega \overline{c_\sigma}(p_{\sigma'}) / \sim$ , consisting of classes of piecewise-smooth loops in  $\overline{c_\sigma}$  based at  $p_{\sigma'}$  for any  $\sigma' \supset \sigma$ , under retracing equivalence.<sup>3</sup>

**Lemma 2.** Every element  $[\gamma] \in \mathcal{P}_\mathcal{C}$  admits a minimal factorization

$$[\gamma] = [\gamma_{v_r}] \cdots [\gamma_{v_2}] \cdot [\gamma_{v_1}],$$

with  $[\gamma_{v_i}] \in \mathcal{P}_{v_i}$  for some  $c_{v_1}, \dots, c_{v_r} \in \mathcal{C}_n$ . Therefore, the path groupoid  $\mathcal{P}_\mathcal{C}$  is in particular generated by the local subgroupoids  $\{\mathcal{P}_v\}_{c_v \in \mathcal{C}_n}$ .

*Proof.* There exist multiple potential factorizations for a given path, but we prescribe a specific one, with a minimality property, as follows. Choose any representative  $\gamma \in [\gamma]$  such that its image intersects the interiors of a finite and *minimal* number of  $n$ -cells  $c_1, \dots, c_r \in \mathcal{C}_n$  (in the sense that there is no subcollection of  $n$ -cells with the same property, although a given  $n$ -cell may appear several times). Then, there exist subintervals  $[a_1, b_1], \dots, [a_{r-1}, b_{r-1}]$  in  $[0, 1]$ , such that

$$\gamma([a_i, b_i]) \subset \overline{c_{v_i}} \cap \overline{c_{v_{i+1}}},$$

while for any other  $t \in [0, 1] \setminus (\cup_{i=1}^{r-1} [a_i, b_i])$ ,  $\gamma(t)$  lies in the interior of one of the previous  $n$ -cells. For every  $i = 1, \dots, r-1$ , choose the minimal subcell  $c_{\sigma_i}$  of  $\overline{c_{v_i}} \cap \overline{c_{v_{i+1}}}$  such that  $\gamma([a_i, b_i]) \subset \overline{c_{\sigma_i}}$ . Then, there exists another path  $\gamma'$ , thinly equivalent to  $\gamma$ , with an additional explicit factorization into  $r$  subpaths

$$\gamma' = \gamma'_r \cdots \gamma'_1,$$

and for each  $i = 1, \dots, r-1$ ,  $\gamma'_i([0, 1]) \subset \overline{c_{v_i}}$ , and  $t(\gamma'_i) = p_{\sigma_i}$ . Letting  $[\gamma_{v_i}] = [\gamma'_i]$ , the claim follows.  $\square$

The previous factorization of elements in  $\mathcal{P}_\mathcal{C}$  into local pieces motivates the introduction of local path families as essential building blocks for the study of subgroupoids in  $\mathcal{P}_\mathcal{C}$ . For any pair of cells  $c_\sigma$  and  $c_{\sigma'}$  in  $\mathcal{C}$  such that  $\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset$ , an  $r$ -dimensional *local path family*  $\mathcal{F}_{\sigma', \sigma}^r$ , of paths  $\gamma^x$  satisfying  $s(\gamma^x) = p_\sigma$ , and  $t(\gamma^x) = p_{\sigma'}$ , is understood as an equivalence class of piecewise-smooth maps<sup>4</sup>

$$f : \overline{\mathbb{D}^r} \times [0, 1] \rightarrow \overline{c_\sigma} \cup \overline{c_{\sigma'}}$$

such that  $f(x, 0) = p_\sigma$  and  $f(x, 1) = p_{\sigma'}$  for any  $x \in \overline{\mathbb{D}^r}$ , under simultaneous retracings, i.e., piecewise-smooth homotopy maps  $H : \overline{\mathbb{D}^r} \times [0, 1]^2 \rightarrow \overline{c_\sigma}$  such that for every fixed  $x \in \overline{\mathbb{D}^r}$ ,  $\gamma_s^x(t) = H(x, s, t)$  defines a thin homotopy of

<sup>3</sup>For any fixed  $\sigma$  and a pair  $\sigma'_1, \sigma'_2 \supset \sigma$ , the loop groups  $\mathcal{L}(\overline{c_\sigma}, p_{\sigma'_1})$  and  $\mathcal{L}(\overline{c_\sigma}, p_{\sigma'_2})$  are isomorphic under the choice of  $\gamma \in \mathcal{P}_\sigma$  such that  $s(\gamma) = p_{\sigma'_1}$  and  $t(\gamma) = p_{\sigma'_2}$ .

<sup>4</sup>The paths in the smooth family are defined by decreeing  $\gamma^x(t) = f(x, t)$  for every fixed  $x \in \overline{\mathbb{D}^r}$ .

paths. Of special relevance will be the cases (i) when  $c_\sigma, c_{\sigma'} \in \mathcal{C}_n$ , and (ii) when  $\overline{c_{\sigma'}} \subset \overline{c_\sigma}$ .

There is a special class of local path families we are interested in, which will be called *cellular path families*, satisfying three crucial properties, and that will be considered exclusively henceforth. The first condition is that, given a pair of cells such that  $\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset$  as before, we will require the path family  $\mathcal{F}_{\sigma'\sigma}$  to satisfy additionally that there is a bijection between  $\mathcal{F}_{\sigma'\sigma}$  and  $\overline{c_\sigma} \cap \overline{c_{\sigma'}}$ ; in particular, it follows that

$$\dim \mathcal{F}_{\sigma'\sigma} = \dim \overline{c_\sigma} \cap \overline{c_{\sigma'}}.$$

Moreover, whenever  $c_{\sigma'_1}, c_{\sigma'_2} \subset \overline{c_\sigma}$  satisfy that  $\overline{c_{\sigma'_1}} \cap \overline{c_{\sigma'_2}} \neq \emptyset$ , we will require the compatibility condition

$$\mathcal{F}_{\sigma'_1\sigma}|_{\overline{c_{\sigma'_1}} \cap \overline{c_{\sigma'_2}}} = \mathcal{F}_{\sigma'_2\sigma}|_{\overline{c_{\sigma'_1}} \cap \overline{c_{\sigma'_2}}},$$

and therefore, it is sufficient to prescribe them as a collection of path families parametrized by  $\mathcal{C} \setminus \mathcal{C}_0$ —all cells  $c_\sigma$  of positive dimension—in terms of the subcollections

$$\mathfrak{F}_\sigma = \{\mathcal{F}_{\sigma'\sigma}\}_{c_{\sigma'} \subset \overline{c_\sigma}}$$

where the thin homotopy classes in the families of each subcollection  $\mathfrak{F}_\sigma$  are represented by a smooth family of paths joining the given baricenter  $p_\sigma \in c_\sigma$  with all the points in  $\partial \overline{c_\sigma}$ . The third and final condition that we will require is that the smooth families of representatives for each  $\mathfrak{F}_\sigma$  are moreover prescribed in terms of a path foliation of  $\overline{c_\sigma} \setminus \{p_\sigma\}$  with  $\partial \overline{c_\sigma}$  as its parameter space. In fact, any such special classes of path families can be constructed if one considers a pair of diffeomorphisms

$$\psi_\sigma : \overline{\mathbb{D}^k} \rightarrow \overline{c_\sigma} \quad \text{and} \quad \psi_{\sigma'} : \overline{\mathbb{D}^{k'}} \rightarrow \overline{c_{\sigma'}}$$

such that  $\psi_{\sigma'}(0) = p_\sigma$  and  $\psi_\sigma(0) = p_{\sigma'}$ . The path  $\gamma_{\sigma'\sigma}^x$  in the family  $\mathcal{F}_{\sigma'\sigma}$ , corresponding to  $x \in \overline{c_\sigma} \cap \overline{c_{\sigma'}}$  is constructed as follows. Let  $\gamma_\sigma^x$  and  $\gamma_{\sigma'}^x$  be the respective images in  $\overline{c_\sigma}$ ,  $\overline{c_{\sigma'}}$  under  $\psi_\sigma$  and  $\psi_{\sigma'}$ , of the linear segments in their unit ball domains, starting from 0, and such that  $t(\gamma_\sigma^x) = t(\gamma_{\sigma'}^x) = x$ . Then,  $\gamma_{\sigma'\sigma}^x = (\gamma_{\sigma'}^x)^{-1} \cdot \gamma_\sigma^x$ . In the special case when  $\overline{c_{\sigma'}} \subset \overline{c_\sigma}$  and  $x = p_{\sigma'}$ , we will simply denote  $\gamma_{\sigma'\sigma}^x$  as  $\gamma_{\sigma'\sigma}$ .

*Remark 4.* The cellular path families defined before satisfy the following fundamental property. For any  $k''$ -subcell  $c_{\sigma''} \subset \overline{c_\sigma} \cap \overline{c_{\sigma'}}$ ,  $k'' > 0$ , consider a choice of cellular path families  $\mathcal{F}_{\sigma'\sigma}$ ,  $\mathcal{F}_{\sigma''\sigma}$ , and  $\mathcal{F}_{\sigma''\sigma'}$ . Whenever  $x \in \overline{c_{\sigma''}}$ , we can induce the local factorization of thin homotopy classes

$$(3.1) \quad [\gamma_{\sigma'\sigma}^x] = [\gamma_{\sigma''\sigma'}^x]^{-1} \cdot [\gamma_{\sigma''\sigma}^x],$$

with  $[\gamma_{\sigma''\sigma}^x] \in \mathcal{F}_{\sigma''\sigma}$ ,  $[\gamma_{\sigma''\sigma'}^x] \in \mathcal{F}_{\sigma''\sigma'}$ , which could be written symbolically as

$$\mathcal{F}_{\sigma'\sigma}|_{\overline{c_{\sigma''}}} = \mathcal{F}_{\sigma''\sigma'}^{-1} \cdot \mathcal{F}_{\sigma''\sigma},$$

and which in particular applies in the case when  $\overline{c_{\sigma''}} \subset \overline{c_{\sigma'}} \subset \overline{c_\sigma}$ . There are several types of collections of fundamental cellular path families, connecting

all base points in the elements of  $\mathcal{C}$ , that will be important for us. The simplest of such is given in terms of a collection

$$\mathfrak{F}_{\min} = \{\mathcal{F}_{vw}\}_{c_v, c_w \in \mathcal{C}_n, \overline{c_v} \cap \overline{c_w} \neq \emptyset},$$

where for every admissible pair  $\{v, w\}$ , the elements in the families  $\mathcal{F}_{vw}$  and  $\mathcal{F}_{wv}$  are related as

$$[\gamma_{wv}^x] = [\gamma_{vw}^x]^{-1}.$$

Any element  $\mathcal{F}_{vw} \in \mathfrak{F}_{\min}$  acquires the structure of an  $(n-1)$ -dimensional cell in  $\mathcal{P}_{\mathcal{C}}$ , which follows from the bijective correspondence with the closed cell  $\overline{c_v} \cap \overline{c_w}$ , and generates an  $(n-1)$ -dimensional topological subgroupoid  $\mathcal{P}_{\min} \subset \mathcal{P}_{\mathcal{C}}$ , the *minimal* cellular path subgroupoid containing the families in the collection  $\mathfrak{F}_{\min}$ . Upon a choice of an additional factorization of the families  $\mathcal{F}_{vw} = \mathcal{F}_{\tau v}^{-1} \cdot \mathcal{F}_{\tau w}$ , where  $\overline{c_{\tau}} = \overline{c_v} \cap \overline{c_w}$ , the minimal path subgroupoid can be further factored into a larger  $(n-1)$ -dimensional topological subgroupoid

$$\mathcal{P}_{\min} \subset \mathcal{P}_{\min'} \subset \mathcal{P}_{\mathcal{C}},$$

namely, the minimal subgroupoid of  $\mathcal{P}_{\mathcal{C}}$  containing the factored families  $\{\mathcal{F}_{\tau v}\}_{c_v \in \mathcal{C}_n, c_{\tau} \in \mathcal{C}_{n-1}, \overline{c_{\tau}} \subset \overline{c_v}}$ . In general, if we consider an arbitrary collection of cellular path families,

$$\mathfrak{F} = \{\mathcal{F}_{\sigma'\sigma}\}_{c_{\sigma}, c_{\sigma'} \in \mathcal{C}', \overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \neq \emptyset}$$

indexed by an arbitrary cellular subcomplex  $\mathcal{C}'$  of  $\mathcal{C}$ , then, there is an induced path subgroupoid  $\mathcal{P}_{\mathfrak{F}} \subset \mathcal{P}_{\mathcal{C}}$  generated by all of the cellular path families in  $\mathfrak{F}$ . In the case when, moreover,  $\mathcal{C}' = \mathcal{C}$ , we will refer to the family and the path subgroupoid induced by  $\mathfrak{F}$  as *complete*, and  $\mathcal{P}_{\mathfrak{F}}$  contains a minimal path subgroupoid  $\mathcal{P}_{\min} \subset \mathcal{P}_{\mathfrak{F}}$ , the *core* of  $\mathcal{P}_{\mathfrak{F}}$ . In particular, in accordance with lemma 2, any element in  $\mathcal{P}_{\mathfrak{F}}$  may be factored as a finite product of elements of the cellular path families  $\mathcal{F}_{\sigma'\sigma}$ . It is important to remark that although for different cells  $c_{\sigma'_1}, c_{\sigma'_2} \subset \overline{c_{\sigma}}$  such that  $\overline{c_{\sigma'_1}} \cap \overline{c_{\sigma'_2}} \neq \emptyset$ , the families  $\mathcal{F}_{\sigma'_1\sigma}$  and  $\mathcal{F}_{\sigma'_2\sigma}$  are disjoint, the subcells in their boundaries corresponding to  $\overline{c_{\sigma'_1}} \cap \overline{c_{\sigma'_2}}$  may be identified by means of multiplication by the “bridging” paths

$$\left[\gamma_{\sigma'_1\sigma}^x\right]^{-1} \cdot \left[\gamma_{\sigma'_2\sigma}^x\right], \quad \left[\gamma_{\sigma'_2\sigma}^x\right]^{-1} \cdot \left[\gamma_{\sigma'_1\sigma}^x\right],$$

where  $x \in \overline{c_{\sigma'_1}} \cap \overline{c_{\sigma'_2}}$ . By definition, all of the cells determined by the family  $\mathfrak{F}$  may be regarded as smooth cells in  $\mathcal{P}_{\mathcal{C}}$ .

We can summarize the construction that we have introduced in remark 4 as the following lemma.

**Lemma 3.** *A triangle-dual cellular decomposition  $\mathcal{C}$  of  $M$ , together with a choice of a collection of cellular path families*

$$\mathfrak{F} = \{\mathcal{F}_{\sigma'\sigma}\}_{c_{\sigma}, c_{\sigma'} \in \mathcal{C}', \overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \neq \emptyset}$$

parametrized by an arbitrary cell subcomplex  $\mathcal{C}' \subset \mathcal{C}$ , defines a topological subgroupoid  $\mathcal{P}_{\mathfrak{F}}$  of the path groupoid  $\mathcal{P}_{\mathcal{C}}$ , generated by closed cells in correspondence with the cellular path families  $\mathcal{F}_{\sigma'\sigma}$ , for  $c_{\sigma}, c_{\sigma'} \in \mathcal{C}'$  such that  $\overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \neq \emptyset$ . The paths  $[\gamma_{\sigma''\sigma}]$  corresponding to  $c_{\sigma''} \in \mathcal{C}_0$  such that  $c_{\sigma''} \subset \overline{c_{\sigma}}$  for  $c_{\sigma} \in \mathcal{C}'$ , determine a discrete subgroupoid  $\mathcal{P}_{\mathfrak{F}}^0 \subset \mathcal{P}_{\mathfrak{F}}$ .

#### 4. CELLULAR PARALLEL TRANSPORT MAPS

**Definition 4.** Let  $G$  be a Lie group. A *smooth cellular parallel transport map*, relative to a choice of cellular decomposition  $\mathcal{C}$  of  $M$ , is a groupoid homomorphism

$$\text{PT}_{\mathcal{C}} : \mathcal{P}_{\mathcal{C}} \rightarrow G,$$

such that, for any choice of local path family  $\mathcal{F}_{\sigma'\sigma}$ , the induced map

$$g_{\sigma'\sigma} : \overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \rightarrow G$$

that results from the evaluation of  $\text{PT}_{\mathcal{C}}$  in  $\mathcal{F}_{\sigma'\sigma}$ , is continuous, and its restriction to any subcell  $c_{\sigma''} \subset \overline{c_{\sigma}} \cap \overline{c_{\sigma'}}$  is smooth. We say that two cellular parallel transport maps  $\text{PT}_{\mathcal{C}}$  and  $\text{PT}'_{\mathcal{C}}$  are equivalent if for any  $c_{\sigma} \in \mathcal{C}$  there is an element  $g_{\sigma} \in G$  such that

$$\text{PT}'_{\mathcal{C}}([\gamma]) = g_{\sigma_2} \text{PT}_{\mathcal{C}}([\gamma]) g_{\sigma_1}^{-1}$$

for any  $[\gamma] \in \mathcal{P}_{\mathcal{C}}$  with  $s(\gamma) = p_{\sigma_1}$ , and  $t(\gamma) = p_{\sigma_2}$ .

*Remark 5.* Technically speaking, there is a slightly more general way to define a smooth cellular parallel transport map, by replacing  $G$  with another groupoid in the defining homomorphism. Namely, consider a groupoid  $G_M$ , which to every point  $x \in M$ , assigns a  $G$ -torsor  $P_x$ , and to every morphism  $x \rightarrow y$ ,  $x, y \in M$ , assigns a  $G$ -equivariant map  $P_x \rightarrow P_y$ , that is, a covariant functor from  $M$ , thought of as a category, to the category of  $G$ -torsors. We could then define a general parallel transport map as a groupoid homomorphism  $\text{PT} : \mathcal{P}_M \rightarrow G_M$ , where  $\mathcal{P}_M$  is the full path groupoid in  $M$ , satisfying an analogous smoothness condition. As we will see later, such homomorphism would allow to define a principal  $G$ -bundle  $P \rightarrow M$ . Moreover, the specialization to definition 4 will provide a collection of trivializations on the fibers over the base points of the elements in  $\mathcal{C}$ .

It follows as a consequence of lemma 2, that a smooth cellular parallel transport map is equivalent to a collection of groupoid homomorphisms

$$\{\text{PT}_{\sigma} : \mathcal{P}_{\sigma} \rightarrow G\}_{c_{\sigma} \in \mathcal{C}},$$

compatible in the sense that for any  $\tau \subset \sigma$  and  $[\gamma] \in \mathcal{P}_{\sigma} \subset \mathcal{P}_{\tau}$ , it follows that  $\text{PT}_{\tau}([\gamma]) = \text{PT}_{\sigma}([\gamma])$ , and satisfying a suitable smoothness condition. We may consider the former, or a collection of the latter, whenever it is more convenient.

*Remark 6.* A cellular parallel transport map  $\text{PT}_{\mathcal{C}}$  allows us to define a collection of principal  $G$ -bundles  $\pi_{\sigma} : P_{\sigma} \rightarrow \overline{c_{\sigma}}$  for every  $c_{\sigma} \in \mathcal{C}$ . Let  $\mathcal{P}_{\sigma}^s$

be the path space of thin homotopy classes of piecewise-smooth paths in  $\overline{c_\sigma}$  whose source is  $p_{\sigma'}$ , for any  $\sigma' \supset \sigma$ . Let

$$(4.1) \quad P_\sigma = \mathcal{P}_\sigma^s \times G / \sim_{\text{PT}}$$

where two pairs  $([\gamma], g)$  and  $([\gamma'], g')$  are related if  $t(\gamma) = t(\gamma')$  and

$$g' = \text{PT}_\sigma \left( [\gamma']^{-1} \cdot [\gamma] \right) g.$$

The projection of a class  $[\gamma, g]$  onto  $M$  is simply defined as  $[\gamma, g] \mapsto t(\gamma)$ , determining a map  $\pi_\sigma : P_\sigma \rightarrow \overline{c_\sigma}$ . Moreover, there is a global right  $G$ -action on  $P_\sigma$ , defined as  $[\gamma, g'] \cdot g = [\gamma, g'g]$ . For every  $\sigma' \supset \sigma$ , there is a special point  $b_{\sigma'} \in \pi_\sigma^{-1}(p_{\sigma'})$ , determined by simply considering the classes  $[p_{\sigma'}, e]$ , where, by a slight abuse of notation,  $[p_{\sigma'}]$  represents the class of the constant path at  $p_{\sigma'}$ . Consequently, there is a identification of the fiber  $\pi_\sigma^{-1}(p_{\sigma'})$  with  $G$ .

*Remark 7.* A smooth and global *cellular trivialization* can be given for each principal  $G$ -bundle  $\pi_\sigma : P_\sigma \rightarrow \overline{c_\sigma}$ . This can be seen by considering, similarly to remark 4, a smooth family of paths  $\mathcal{F}_\sigma^s \subset \mathcal{P}_\sigma^s$  with source  $p_\sigma$ , in bijective correspondence with  $\overline{c_\sigma}$ , with bijection determined by the path target map  $\gamma_\sigma^x \mapsto t(\gamma_\sigma^x) = x$  (for instance, by taking any diffeomorphism  $\psi_\sigma : \overline{\mathbb{D}^k} \rightarrow \overline{c_\sigma}$ , for every  $c_\sigma \in \mathcal{C}_k$ ,  $1 \leq k \leq n$ , such that  $\psi_\sigma(0) = p_\sigma$ , and the collection of paths  $\{\gamma_\sigma^x\}$  in  $\mathcal{P}_\sigma^s$ , consisting of the images of linear segments in  $\overline{\mathbb{D}^k}$  with source at 0). This way, we get a bijection

$$\Psi_\sigma : \overline{c_\sigma} \times G \rightarrow P_\sigma, \quad (x, g) \mapsto [\gamma_\sigma^x, g],$$

which defines a smooth structure of manifold with boundary on  $P_\sigma$ , and a trivialization as a smooth principal  $G$ -bundle over  $\overline{c_\sigma}$ .

Since whenever  $\sigma \supset \tau$ , we have that  $\mathcal{P}_\sigma^s \subset \mathcal{P}_\tau^s$ , we can construct a smooth bijection  $P_\sigma \mapsto P_\tau|_{\overline{c_\sigma}}$  by restriction, and therefore, the collection of principal  $G$ -bundles  $\{P_\sigma\}_{c_\sigma \in \mathcal{C}}$  can be glued into a single bundle  $\pi : P \rightarrow M$ , which could also be constructed as a quotient similar to (4.1), in terms of the full path space  $\mathcal{P}_\mathcal{C}^s$  of all thin homotopy equivalence classes of piecewise-smooth paths with source an arbitrary  $p_\sigma$  in  $M$ , and projection  $\pi$  defined in a similar way. As before, such bundle would come with a preferred set of points  $\{b_\sigma \in \pi^{-1}(p_\sigma)\}_{c_\sigma \in \mathcal{C}}$ , which we will denote by  $\mathcal{E}_\mathcal{C}$ . Thus,

$$\pi(\mathcal{E}_\mathcal{C}) = \mathcal{B}_\mathcal{C}.$$

It is possible to give a more straightforward construction of the principal  $G$ -bundle  $P$  by means of local trivializations and transition functions. Consider an arbitrary complete collection of path families  $\{\mathcal{F}_\sigma^s\}_{c_\sigma \in \mathcal{C}}$ , where  $\mathcal{F}_\sigma^s = \{\gamma_\sigma^x : x \in \overline{c_\sigma}\}$  as in remark 7. Then, whenever  $\tau \subset \sigma$ , the identity

$$[\gamma_\tau^x, g] = [\gamma_\sigma^x, g']$$

for any  $x \in \overline{c_\sigma} \subset \overline{c_\tau}$ , together with the corresponding trivializations, allows us to express  $g' = g_{\sigma\tau}(x)g$  for some  $g_{\sigma\tau}(x) \in G$ . We can explicitly write

$$(4.2) \quad g_{\sigma\tau}(x) = \text{PT}_{\mathcal{C}} \left( [\gamma_\sigma^x]^{-1} \cdot [\gamma_\tau^x] \right).$$

Moreover, if the collection of path families  $\{\mathcal{F}_\sigma^s\}$  are changed to any other given one, the elements  $g_{\sigma\tau}(x)$  would transform as

$$g_{\sigma\tau}(x) \mapsto g_\sigma(x)g_{\sigma\tau}(x)g_\tau(x)^{-1},$$

for some well-defined smooth functions  $g_\sigma : \overline{c_\sigma} \rightarrow G$  and  $(g_\tau : \overline{c_\tau} \rightarrow G)|_{\overline{c_\sigma}}$ .

**Definition 5.** The *glueing maps* induced from a cellular parallel transport map  $\text{PT}_{\mathcal{C}}$  and a complete collection of cellular path families  $\mathfrak{F}$ , for a flag  $\overline{c_\sigma} \subset \overline{c_\tau}$  in  $\mathcal{C}$ ,<sup>5</sup> are the  $g_{\sigma\tau} : \overline{c_\sigma} \rightarrow G$  defined above. Clearly, the glueing maps satisfy the factorization property

$$(4.3) \quad (g_{\sigma\tau}|_{\overline{c_{\sigma'}}}) = g_{\sigma'\sigma}^{-1} \cdot g_{\sigma'\tau}$$

whenever  $\tau \subset \sigma \subset \sigma'$ . The *clutching maps*, or *skeletal transition functions*, for a minimal collection  $\mathfrak{F}_{\min}$  (remark 4), are the maps of the form

$$(4.4) \quad h_{vw}(x) = \text{PT}_{\mathcal{C}}([\gamma_{vw}^x]) = g_{\tau v}^{-1} \cdot g_{\tau w}, \quad [\gamma_{vw}^x] \in \mathcal{F}_{vw}$$

for any  $x \in \overline{c_v} \cap \overline{c_w}$ , where  $v, w \subset \tau$ , for some  $c_\tau \in \mathcal{C}_{n-1}$ .

Clearly, the clutching maps satisfy the relation  $h_{wv} = h_{vw}^{-1}$ . Moreover, when cyclically oriented triples are taken into account (remarks 2 and 3), whenever  $\overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} \neq \emptyset$ , and (2.2) is satisfied for  $c_\sigma \in \mathcal{C}_{n-2}$ , these maps satisfy the cocycle condition

$$h_{v_1 v_2} h_{v_2 v_3} h_{v_3 v_1}|_{\overline{c_\sigma}} = e.$$

As we will see in section 5, the maps  $\{h_{vw} : \overline{c_\tau} \rightarrow G\}_{v,w \subset \tau, c_\tau \in \mathcal{C}_{n-1}}$  play the role of transition functions for the principal bundle  $P$ . Indeed, it is easy to verify that they transform as transition functions under equivalence of cellular parallel transport maps. The latter is stated in precise terms in the next proposition.

*Remark 8.* A *horizontal lift to  $P$* , for every path  $\gamma : [0, 1] \rightarrow M$  and a choice of initial condition  $[\gamma', g] \in \pi^{-1}(\gamma(0))$  can be canonically constructed from the cellular parallel transport map  $\text{PT}_{\mathcal{C}}$ . Namely, horizontal lifts can first be defined if we take any path  $\gamma : [0, 1] \rightarrow M$  and an initial condition  $[\gamma', g]$ , where  $[\gamma'] \in \mathcal{P}_{\mathcal{C}}^s$ . Then by the independence under reparametrization, a path  $\lambda(\gamma) : [0, 1] \rightarrow P_\sigma$  can be defined as

$$[\lambda(\gamma)](\epsilon) = [(\gamma \cdot \gamma')|_{[0, (\epsilon+1)/2]}, g],$$

for  $\epsilon \in [0, 1]$ , with source  $s(\lambda(\gamma)) = [\gamma', g]$ , satisfying  $\pi \circ \lambda(\gamma)(\epsilon) = \gamma(\epsilon)$ . Horizontal lifts may also be constructed as an iteration of local steps, if we recall the factorization of a class of piecewise smooth paths in  $M$  in lemma

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<sup>5</sup>When complete cellular path families are considered, it is enough to define the glueing maps over flags of length 2 in  $\mathcal{C}$ , as any other glueing map could be induced from those.

2. The local cellular horizontal lifts on each  $\overline{c_{\sigma_i}}$ ,  $i = 1, \dots, r$ , together with an iteration of subsequent initial conditions, determines the full horizontal lift of  $\gamma$  to  $P$ .

**Proposition 1.** *Let us fix a triangle-dual cellular decomposition  $\mathcal{C}$  in  $M$ .*

- (i) *If two different cellular parallel transport maps  $\text{PT}_{\mathcal{C}}$  and  $\text{PT}'_{\mathcal{C}}$  are equivalent, then there is a set of constants  $\{g_{\sigma}\}_{c_{\sigma} \in \mathcal{C}}$ , in such a way that the glueing maps  $\{g_{\sigma'\sigma}\}_{\overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \neq \emptyset}$  and  $\{g'_{\sigma'\sigma}\}_{\overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \neq \emptyset}$ , induced from a fixed choice of complete cellular path families  $\mathfrak{F} = \{\mathcal{F}_{\sigma'\sigma}\}$ , are related as*

$$g'_{\sigma'\sigma}(x) = g_{\sigma'} \cdot g_{\sigma'\sigma}(x) \cdot g_{\sigma}^{-1}.$$

- (ii) *If the choice of  $\mathfrak{F}$  is changed, then there is a set of smooth maps  $\{g_{\sigma}(x) : \overline{c_{\sigma}} \rightarrow G\}_{c_{\sigma} \in \mathcal{C}}$ , in such a way that the maps  $\{g_{\sigma'\sigma}\}_{\overline{c_{\sigma}} \cap \overline{c_{\sigma'}} \neq \emptyset}$  induced from  $\text{PT}_{\mathcal{C}}$  transform pointwise as*

$$g_{\sigma'\sigma}(x) \mapsto g_{\sigma'}(x) g_{\sigma'\sigma}(x) g_{\sigma}(x)^{-1}.$$

*Proof.* (i) is a straightforward consequence of definition 4. Namely, the value of  $g_{\sigma'\sigma}(x)$  (and similarly for the value  $g'_{\sigma'\sigma}(x)$ ), for an arbitrary  $x \in \overline{c_{\sigma}} \cap \overline{c_{\sigma'}}$ , is equal to  $\text{PT}_{\mathcal{C}}([\gamma_{\sigma'\sigma}^x])$ . (ii) has been discussed already. It should be remarked that the proposition applies in particular to the clutching maps  $h_{vw} = g_{vw}$ .  $\square$

In fact, it will turn out that only the homotopy type of the glueing maps  $\{g_{\sigma'\sigma}\}$ , in a suitable sense, will be relevant to determine an equivalence class of principal  $G$ -bundles on  $M$ . Such notion of homotopical *cellular bundle data*, characterizing a principal  $G$ -bundle with trivializations over  $\mathcal{C}$ , up to equivalence, is axiomatized in definition 10.

*Remark 9.* It is also possible to construct an honest system of transition functions for the principal bundle  $\pi : P \rightarrow M$  if each of the local bundles  $P_{\sigma}$  is constructed instead over the corresponding open set  $\mathcal{U}_{\sigma}$ , by extending the spaces  $\mathcal{P}_{\sigma}^s$  to consist of paths belonging to  $\mathcal{U}_{\sigma}$  (and not only to  $\overline{c_{\sigma}}$ ), and repeating the previous constructions verbatim.

Finally, we remark that the notion of a cellular parallel transport map, which we have axiomatized before, together with the horizontal lift property described in remark 8, implies the existence of an equivalence class of smooth connections in the principal  $G$ -bundle  $P$ , under the relation determined by the action of the group of bundle automorphisms covering the identity map in  $M$ , and acting as the identity over the fibers  $\{\pi^{-1}(p_{\sigma})\}_{c_{\sigma} \in \mathcal{C}}$ , which we will refer to as *restricted gauge transformations*. The group  $\mathcal{G}_{P,*}$  of restricted gauge transformations is a finite-codimension normal subgroup of the full gauge group  $\mathcal{G}_P$  of  $P$ .

We summarize our findings in the form of a theorem. We have provided a rigorous proof of theorem 2 in [23], which is a cellular analog to Barrett's reconstruction theorem [3]. Reformulated in our terminology, the statement is the following.



**Theorem 1** (Reconstruction theorem). *Let  $M$  be an oriented  $n$ -manifold, equipped with a triangle-dual cellular decomposition  $\mathcal{C}$ , and  $G$  a Lie group. There is a bijective correspondence*

$$\left\{ \begin{array}{l} G\text{-valued smooth} \\ \text{cellular parallel} \\ \text{transport maps } \text{PT}_{\mathcal{C}} \\ \text{in } M, \text{ up to} \\ \text{equivalence} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{smooth principal } G\text{-bundles } P \rightarrow M \\ \text{with a choice of fiber points } \mathcal{E}_{\mathcal{C}}, \text{ up} \\ \text{to equivalence, and a smooth} \\ \text{connection up to restricted gauge} \\ \text{equivalence} \end{array} \right\}$$

and moreover, for any smooth cellular parallel transport map  $\text{PT}_{\mathcal{C}}$  with a principal  $G$ -bundle  $P$  representing its induced equivalence class  $\{P\}$ , there is an injective correspondence

$$\left\{ \begin{array}{l} \text{Path families } \mathcal{F}_{\sigma}^s, \\ \text{for } c_{\sigma} \in \mathcal{C} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{Local trivializations of} \\ P \text{ over } \overline{c_{\sigma}}, \text{ for } c_{\sigma} \in \mathcal{C} \end{array} \right\}$$

We will finish the present section with definition 8, the main concept on which the rest of the article will be built on. Such definition is cemented in the following couple of definitions regarding discrete topological structures in the path groupoid  $\mathcal{P}_{\mathcal{C}}$ , and provides a justification for the insufficiency of the standard lattice gauge theory (LGT) data to capture the global topology of a principal  $G$ -bundle.

**Definition 6.** A *cellular network*  $\Gamma$  in a triangle-dual cellular decomposition  $\mathcal{C}$  of  $M$ , relative to a collection of cell base points  $\mathcal{B}_{\mathcal{C}} = \{p_{\tau} \in c_{\tau}\}_{c_{\tau} \in \mathcal{C}}$ , is a collection of nonintersecting paths  $\gamma_{\sigma\tau} \subset \overline{c_{\tau}}$  for every  $c_{\tau} \in \mathcal{C}$ , and every 0-cell  $c_{\sigma} = p_{\sigma} \in \partial \overline{c_{\tau}}$ , joining  $p_{\tau}$  and  $p_{\sigma}$ . We will denote by  $\mathcal{P}_{\Gamma}^0$  the discrete groupoid of the path groupoid  $\mathcal{P}_{\mathcal{C}}$  generated by a cellular network  $\Gamma$ , or by  $\mathcal{P}_{\mathfrak{F}}^0$  if  $\Gamma$  is induced by a complete collection of cellular path families  $\mathfrak{F}$ .

**Definition 7.** A *standard lattice gauge field*, relative to a cellular parallel transport map  $\text{PT}_{\mathcal{C}}$  and a cellular network  $\Gamma = \{\gamma_{\sigma\tau}\}_{p_{\sigma} \in \partial \overline{c_{\tau}}, c_{\tau} \in \mathcal{C}}$ , is an assignment of a group element  $\text{PT}_{\mathcal{C}}([\gamma_{\sigma\tau}])$  for every path  $\gamma_{\sigma\tau}$ , or equivalently, a groupoid homomorphism  $\text{PT}_{\Gamma}^0 : \mathcal{P}_{\Gamma}^0 \rightarrow G$ .

**Definition 8.** Let  $M$  be an oriented  $n$ -dimensional manifold,  $n \geq 2$ , with a triangle-dual cellular decomposition  $\mathcal{C}$ , a complete collection  $\mathfrak{F}$  of cellular path families (remark 4), and its induced path subgroupoid  $\mathcal{P}_{\mathfrak{F}}$ . By an *extended lattice gauge field*, we mean an equivalence class of cellular parallel transport maps  $\{\text{PT}_{\mathcal{C}}\}_{\mathfrak{F}}$ , where  $\text{PT}_{\mathcal{C}} \sim \text{PT}'_{\mathcal{C}}$  if their restrictions to the path subgroupoid  $\mathcal{P}_{\mathfrak{F}} \subset \mathcal{P}_{\mathcal{C}}$  are homotopic, relative to a fixed choice of standard lattice gauge field over its discrete subgroupoid  $\mathcal{P}_{\mathfrak{F}}^0$ .

*Remark 10.* In fact, the definition of an extended lattice gauge field does not require to completely fix a specific collection of cellular path families  $\mathfrak{F}$ , since any other collection  $\mathfrak{F}'$  whose induced discrete subgroupoid  $\mathcal{P}_{\mathfrak{F}'}^0$  coincides with  $\mathcal{P}_{\mathfrak{F}}^0$  would determine the same homotopy class of cellular parallel transport maps, i.e.,  $\{\text{PT}_{\mathcal{C}}\}_{\mathfrak{F}} = \{\text{PT}_{\mathcal{C}}\}_{\mathfrak{F}'}$ . In such case, the path subgroupoids  $\mathcal{P}_{\mathfrak{F}}$  and  $\mathcal{P}_{\mathfrak{F}'}$  are smoothly homotopic, and by definition, there



is a 1-1 correspondence between smooth homotopies of subgroupoids  $\mathcal{P}_{\mathfrak{F}}$  fixing  $\mathcal{P}_{\mathfrak{F}}^0$  and smooth homotopies of collections of cellular path families  $\mathfrak{F}$  with common, fixed paths connecting base points to 0-boundary cells. Since there is only one such homotopy class for every pair of intersecting cells  $c_\sigma, c_{\sigma'} \in \mathcal{C}$ , we could think of the homotopy class of path subgroupoids  $\{\mathcal{P}_{\mathfrak{F}}\}$ , relative to a fixed choice of discrete subgroupoid  $\mathcal{P}_\Gamma^0$ , as an intrinsic object associated to  $\mathcal{C}$  and  $\mathcal{P}_\Gamma^0$ . Hence, an extended gauge field is also an intrinsic object, depending only of  $\mathcal{C}$  and a choice of standard lattice gauge field  $\text{PT}_\Gamma^0$ .

## 5. EXTENDED LATTICE GAUGE FIELDS AND PRINCIPAL BUNDLES

Let us now assume for simplicity that  $G$  is a *connected* Lie group. The extra complications of the general case are easy to sort out, as the connected component of the identity  $G_0$  is normal in  $G$ , and  $\pi_0(G, e) \cong G/G_0$ .

We have seen that a special characteristic feature that distinguishes a triangle-dual cellular decomposition  $\mathcal{C}$  on an oriented  $n$ -manifold  $M$ , is that every triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathcal{C}_n$  intersecting nontrivially, determines a triple  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\} \subset \mathcal{C}_{n-1}$  and a unique  $c_\sigma \in \mathcal{C}_{n-2}$ , together with a collection of gapless flags of oriented cells

$$(5.1) \quad \begin{array}{ccccc} & & \overline{c_{v_1}} & & \\ & \subset & & \supset & \\ & \overline{c_{\tau_3}} & \supset & \overline{c_\sigma} & \subset & \overline{c_{\tau_2}} \\ & \supset & & \cap & & \supset \\ \overline{c_{v_2}} & & \supset & \overline{c_{\tau_1}} & \subset & \overline{c_{v_3}} \end{array}$$

a structure that is invariant under index permutations (remarks 2 and 3), and conversely, such  $c_\sigma$ , together with an additional choice of orientation, determines the triples  $\{c_{v_1}, c_{v_2}, c_{v_3}\}$  and  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$ , up to permutations of the indices. An orientation of  $c_\sigma$  corresponds to a choice of gapless flag in the collection, up to the action of the alternating group  $A_3$  of cyclic permutations. An arbitrary collection of clutching maps, relative to  $\mathcal{C}$ ,

$$\{h_{vw}\}_{\overline{c_v} \cap \overline{c_w} \neq \emptyset, c_v, c_w \in \mathcal{C}_n}$$

must satisfy a similar invariance property under a suitably defined  $S_3$ -action, for every triple of  $n$ -cells as above, when the corresponding triple of maps are restricted to their common  $(n-2)$ -cell closure  $\overline{c_\sigma}$ . Such  $S_3$ -action can be described as follows. There is an action of  $S_3$  on  $G \times G \times G$ , prescribed on a choice of two generators in the following way: for the (even) 3-cycle  $(123)$ ,  $(123) \cdot (g_1, g_2, g_3) = (g_3, g_1, g_2)$ , and for the (odd) transposition  $(12)$ ,  $(12) \cdot (g_1, g_2, g_3) = (g_2^{-1}, g_1^{-1}, g_3^{-1})$ . Now, consider the multiplication map

$$T_G : G \times G \times G \rightarrow G, \quad (g_1, g_2, g_3) \mapsto g_1 g_2 g_3,$$

let  $V_G = T_G^{-1}(e)$ , and denote by  $\text{pr}_i$  the projections from  $G \times G \times G$  into the  $i$ th factor. In particular, it readily follows that  $V_G$  is invariant under the  $S_3$ -action defined before, and that  $V_G \cong G \times G$  under the projections  $\text{pr}_i \times \text{pr}_j$

to any pair of distinct  $i, j \in \{1, 2, 3\}$ . The previous action is faithful, and we will refer to it as the *triadic* action for the group  $G$ .

Consider now any triple of  $n$ -cells  $\{c_{v_1}, c_{v_2}, c_{v_3}\}$  as before, together with its collection of clutching maps, which is parametrized by the ordered triples  $(ijk)$ ,  $i, j, k \in \{1, 2, 3\}$ ,  $i \neq j$ ,  $k \neq i, j$ . Define the maps

$$\mathbf{h}_\sigma^{ijk} := (h_{v_i v_j} |_{\overline{c_\sigma}}) \times (h_{v_j v_k} |_{\overline{c_\sigma}}) \times (h_{v_k v_i} |_{\overline{c_\sigma}}) : \overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma} \rightarrow G \times G \times G$$

which satisfy that their restriction to the diagonal  $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$  lies in  $V_G$ . The fact that the labeling in the triple of  $n$ -cells is actually arbitrary is equivalent to the fact that the  $S_3$ -action on the set of maps

$$\{\mathbf{h}_\sigma^{ijk}\}$$

given by index permutations, is determined by postcomposition with the triadic action on their respective images. Thus, we can, and we will, fix an arbitrary labeling on each intersecting triple of  $n$ -cells.

**Definition 9.** 1 A continuous map  $h : \overline{c_\tau} \rightarrow G$  is said to be *cellularly-smooth* if its restriction to any subcell  $c_\sigma \subset \overline{c_\tau}$  is smooth. For any triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathcal{C}_n$  as in (5.1), we say that two triples of cellularly-smooth maps

$$(h_{v_1 v_2}, h_{v_2 v_3}, h_{v_3 v_1}), (h'_{v_1 v_2}, h'_{v_2 v_3}, h'_{v_3 v_1}) : \overline{c_{\tau_3}} \times \overline{c_{\tau_1}} \times \overline{c_{\tau_2}} \rightarrow G \times G \times G$$

whose restriction to the diagonal  $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$  lies in  $V_G$ , are *cellularly equivalent* if there is a homotopy of cellularly-smooth maps

$$(h_{v_1 v_2}(t), h_{v_2 v_3}(t), h_{v_3 v_1}(t)) : \overline{c_{\tau_3}} \times \overline{c_{\tau_1}} \times \overline{c_{\tau_2}} \times [0, 1] \rightarrow G \times G \times G$$

between them, whose restriction to  $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$  lies in  $V_G$  for all  $t \in (0, 1)$ .

The following definition is the cornerstone that allows us to recast the notion of an equivalence class of principal  $G$ -bundles in terms of cellular homotopies.

**Definition 10.** Let  $M$  be an oriented  $n$ -dimensional manifold,  $n \geq 2$ , and  $\mathcal{C}$  a triangle-dual cellular decomposition of it. By a choice of *homotopical cellular bundle data*  $\mathcal{D}$ , relative to  $\mathcal{C}$ , we mean an equivalence class of collections of cellularly smooth maps

$$[\{h_{vw} : \overline{c_\tau} \rightarrow G\}_{v,w \subset \tau, c_\tau \in \mathcal{C}_{n-1}}],$$

such that for each for each triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathcal{C}_n$  intersecting nontrivially, the restriction  $h_{v_1 v_2} \times h_{v_2 v_3} \times h_{v_3 v_1} |_{\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})}$  lies in  $V_G$ . Two collections are equivalent if for any triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathcal{C}_n$  intersecting nontrivially, the corresponding triples of maps are cellularly equivalent.

*Remark 11.* We have excluded the case  $n = 1$  in definition 10, since the notion of bundle data is vacuous in such case. Such omission is irrelevant, since principal bundles are always trivial over any 1-dimensional smooth manifold, at least when  $G$  is connected.

The homotopical data of definition 10 is equivalent to an equivalence class of principal  $G$ -bundles: a proof of this fact is provided in theorem 5 in appendix B. The relevance of this characterization of an equivalence class of principal  $G$ -bundles comes from the fact that it is suited to the study of the cellular parallel transport maps of section 4, leading us to define a finer set of homotopical data, equivalent to the extended lattice gauge theory (ELGT) data of definition 8, giving rise to sufficient conditions to capture the homotopy type of all cellular parallel transport maps yielding isomorphic principal  $G$ -bundles. Such data is presented in theorem 2 below, and is the result of dissecting a homotopy class of collections of clutching maps, relative to a fixed set of values over all 0-cells in  $\mathcal{C}$ , and which also determine a well-defined collection of homotopical cellular bundle data. The missing step to establish such characterization is the glueing mechanism, which we now describe in lemma 4. For a finite subset  $Y \subset X$ , where  $X = S^k$  or  $\mathbb{D}^k$ , and a map  $g_0 : Y \rightarrow G$ , let  $[X, G, g_0]$  denote the space of homotopy classes of piecewise smooth maps  $g : X \rightarrow G$  such that  $g|_Y = g_0$ .

**Lemma 4.** *Let  $\mathcal{C}$  be a triangle-dual cellular decomposition of the  $k$ -sphere  $S^k$  and fix a map  $g_0 : \mathcal{C}_0 \rightarrow G$ . If any  $c_\tau \in \mathcal{C}_k$  has assigned a homotopy class of cellularly smooth maps  $[g_\tau] \in [\overline{c_\tau}, G, g_0|_{\overline{c_\tau}}]$ , such that for any  $c_{\tau_1}, c_{\tau_2} \in \mathcal{C}_k$  with  $\overline{c_\sigma} = \overline{c_{\tau_1}} \cap \overline{c_{\tau_2}} \neq \emptyset$ , we have that  $[g_{\tau_1}|_{\overline{c_\sigma}}] = [g_{\tau_2}|_{\overline{c_\sigma}}]$  as homotopy classes in  $[\overline{c_\sigma}, G, g_0|_{\overline{c_\sigma}}]$ , then the classes  $\{[g_\tau]\}_{c_\tau \in \mathcal{C}_k}$  can be glued into a well-defined element  $[g] \in [S^k, G, g_0]$ .*

*Proof.* Upon introducing the glueing mechanism, which is the fundamental step to integrate neighboring classes of cellularly equivalent maps to a new class over the union of their domains, and then repeating it as many times as necessary, it is possible to obtain an honest homotopy class in  $\pi_k(G, g_\tau(c_{\sigma''}))$ . Such procedure will finish after a finite number of steps.

Consider an arbitrary  $k$ -cell, which we denote  $c_{\tau_1}$ , together with all  $k$ -cells neighboring  $c_{\tau_1}$ , that is, the  $c_{\tau'} \in \mathcal{C}_k$  such that  $\overline{c_{\tau_1}} \cap \overline{c_{\tau'}} \neq \emptyset$ , which can also be labeled as  $c_{\tau_2}, \dots, c_{\tau_{m_1}}$ . The first step of the glueing procedure is to construct a class of cellularly equivalent maps defined over the closed set  $\cup_{i=1}^{m_1} \overline{c_{\tau_i}}$ . Such set is either homeomorphic to the closed disk  $\mathbb{D}^k$ , to  $S^k$  minus a finite number of open disks, or to  $S^k$ , and has an induced orientation. We will consider each case, as the first one eventually leads to the last two.

Let us first assume that  $\cup_{i=1}^{m_1} \overline{c_{\tau_i}}$  is a closed disk in  $S^k$ . By hypothesis, for any pair  $\{c_{\tau_1}, c_{\tau_i}\}$  with  $\overline{c_{\tau_1}} \cap \overline{c_{\tau_i}} = \overline{c_{\sigma_{1i}}}$ , the induced classes  $[g_{\tau_1}|_{\overline{c_{\sigma_{1i}}}}]$  and  $[g_{\tau_i}|_{\overline{c_{\sigma_{1i}}}}]$  coincide. Since there are no local obstructions over multiple intersections, we may choose representatives such that  $g_{\tau_1}|_{c_{\tau_{1i}}} = g_{\tau_i}|_{c_{\tau_{1i}}}$ , and moreover, we can apply the same principle to all intersections  $\overline{c_{\tau_{ij}}} = \overline{c_{\tau_i}} \cap \overline{c_{\tau_j}}$ ,  $2 \leq i, j \leq m_1$ , we may also assume that  $g_{\tau_i}|_{c_{\tau_{ij}}} = g_{\tau_j}|_{c_{\tau_{ij}}}$ . Hence, altogether, the representatives  $g_{\tau_1}, \dots, g_{\tau_{m_1}}$  glue to define a piecewise-smooth function on  $\cup_{i=1}^{m_1} \overline{c_{\tau_i}}$ , and hence induce a class of cellularly equivalent maps on  $\cup_{i=1}^{m_1} \overline{c_{\tau_i}}$ . We now add all  $k$ -cells  $c_{\tau''}$  intersecting nontrivially with  $\cup_{i=1}^{m_1} \overline{c_{\tau_i}}$ , and repeat

the previous procedure, until the complementary  $k$ -cells do not intersect pairwise. Then, the resulting set must be  $S^k$  minus a finite number of open disks. Call the remaining cells  $c_{\tau_{f_1}}, \dots, c_{\tau_{f_l}}$ . Upon ordering the closures of  $(k-1)$ -cells  $c_\sigma$  in the boundary of each  $\overline{c_{\tau_{f_j}}}$ , there is a representative  $g_{\tau_{f_j}}$  in its class whose values at each  $\overline{c_\sigma}$  coincide with the boundary values of the previously constructed  $g$ . This way, we obtain a piecewise-smooth map  $g : S^k \rightarrow G$ , up to piecewise-smooth homotopy fixing the values at every 0-cell, whose restriction to any  $k$ -cell recovers the starting homotopy classes.

It remains to check that the previous procedure is independent of the choice of  $k$ -cell in  $S^k$  at every step. Repeat the procedure with any other choice of  $k$ -cells at every step, and call  $g'$  any map constructed in such a way. Since  $g'$  attains the same values than  $g$  at any 0-cell, and by hypothesis, the restriction of  $g'g^{-1} : S^k \rightarrow G$  to any of the  $k$ -cells of  $\mathcal{C}$  would determine a trivial cellular homotopy class, it follows that the homotopy class of  $g'g^{-1}$ , as an element in  $\pi_k(G, e)$ , must be trivial. Then, in particular, it follows that  $[g'] = [g]$  as elements in  $[S^k, G, g_0]$ .  $\square$

*Remark 12.* Let  $c_\sigma$  be a  $k$ -cell in  $M$ , and  $g_0 : \mathcal{C}_0|_{\overline{c_\sigma}} \rightarrow G$  some fixed map. If  $c_{\sigma'} \subset \overline{c_\sigma}$  is some subcell, then every element in  $[\overline{c_\sigma}, G, g_0]$  induces an element in  $[\overline{c_{\sigma'}}, G, g_0|_{\overline{c_{\sigma'}}}]$ . Hence, with the aid of lemma 4, we can give a recursive description of the space  $[\overline{c_\sigma}, G, g_0]$ , in terms of two types of data. The first type is given in terms of the subspace  $\mathcal{H}_\sigma$  of

$$\bigsqcup_{c_{\sigma'} \subset \overline{c_\sigma}} [\overline{c_{\sigma'}}, G, g_0|_{\overline{c_{\sigma'}}}]$$

consisting of elements subject to the condition that over any  $\partial\overline{c_{\sigma'}}$ , the glueing of homotopy classes from lemma 4 is an element in  $[S^{k'}, G, g_0|_{\partial\overline{c_{\sigma'}}}]$  that is trivial when identified with the free homotopy class that contains it, i.e. every representative is also homotopic to a constant map, if one forgets the fixed values at 0-cells. We will call  $\mathcal{H}_\sigma$  the space of *boundary homotopy data*. This way, we get a surjection

$$\text{pr}_\sigma : [\overline{c_\sigma}, G, g_0] \rightarrow \mathcal{H}_\sigma,$$

whose fibers consist of all homotopy extension classes from  $\partial\overline{c_\sigma}$  to  $\overline{c_\sigma}$ , for any given choice of boundary homotopy data. The description of such fibers is done in the next lemma, necessary for the proof of theorem 2. The first part of it is a standard result, which can be used to prove the abelian nature of the homotopy groups of  $G$ , but we have included it here for the sake of completeness.

**Lemma 5.** *The homotopy groups  $\pi_k(G, e)$  can be realized as the sets of homotopy classes of maps  $g : \mathbb{D}^k \rightarrow G$  such that  $g|_{\partial\mathbb{D}^k} = e$ , with product induced from pointwise multiplication of maps. For any  $k$ -cell  $c_\sigma$  and any map  $g_0 : \mathcal{C}_0|_{\overline{c_\sigma}} \rightarrow G$ , there is a free action of  $\pi_k(G, e)$  on  $[\overline{c_\sigma}, G, g_0]$  defined equivalently in terms of left or right pointwise multiplication, whose orbits are the fibers of  $\text{pr}_\sigma$ .*

*Proof.* Let  $x_1, \dots, x_k$  be cartesian coordinates in  $\mathbb{R}^k$ . The standard realization of the multiplication of two classes  $[g], [g'] \in \pi_k(G, e)$  can be described in terms of the choice of a pair of diffeomorphisms  $\psi_{\pm} : \mathbb{D}_{\pm}^k \rightarrow \mathbb{D}^k$ , where  $\mathbb{D}_+^k$  (resp.  $\mathbb{D}_-^k$ ) is the set of points in  $\mathbb{D}^k$  such that  $x_k > 0$  (resp.  $x_k < 0$ ), by letting  $[g] * [g']$  be the class of maps  $g * g'$  such that

$$g * g'|_{\mathbb{D}_+^k} = g \circ \psi_+ \quad \text{and} \quad g * g'|_{\mathbb{D}_-^k} = g' \circ \psi_-$$

for any pair of representatives  $g, g'$ . Upon the choice of a pair of 1-parameter families of open cells  $\mathbb{D}_{\pm}^k(t) \subset \mathbb{D}^k$  such that

$$\mathbb{D}_{\pm}^k(0) = \mathbb{D}_{\pm}^k \quad \text{and} \quad \mathbb{D}_{\pm}^k(1) = \mathbb{D}^k,$$

and complemented with a pair of 1-parameter families of diffeomorphisms  $\psi_{\pm}^t : \mathbb{D}_{\pm}^k(t) \rightarrow \mathbb{D}^k$  such that  $\psi_{\pm}^0 = \psi_{\pm}$  and  $\psi_{\pm}^1 = \text{Id}$ , we can define a based homotopy between any given representative  $g * g'$  and the pointwise product  $gg' : \mathbb{D}^k \rightarrow G$  by letting  $g * g'(t)$  to be equal to

$$\begin{cases} g \circ \psi_+^t & \text{on } \mathbb{D}_+^k(t) \setminus \mathbb{D}_+^k(t) \cap \mathbb{D}_-^k(t), \\ g' \circ \psi_-^t & \text{on } \mathbb{D}_-^k(t) \setminus \mathbb{D}_-^k(t) \cap \mathbb{D}_+^k(t), \\ (g \circ \psi_+^t)(g' \circ \psi_-^t) & \text{on } \mathbb{D}_+^k(t) \cap \mathbb{D}_-^k(t). \end{cases}$$

A similar argument shows the existence of a homotopy between  $g * g'$  and  $g'g$ . The natural actions of  $\pi_k(G, e)$  on  $[\overline{c_{\sigma}}, G, g_0]$  by pointwise left and right multiplication are equivalent by the previous homotopy argument, and are clearly free, since given any  $[g] \in [\overline{c_{\sigma}}, G, g_0]$ , and  $[f] \in \pi_k(G, e)$ , the classes  $[fg] = [gf]$  and  $[g]$  coincide if and only if  $[f]$  is the identity in  $\pi_k(G, e)$ . It is also clear that it preserves the fibers of  $\text{pr}_{\sigma}$ , and that the induced  $\pi_k(G, e)$ -action on any given fiber of  $\text{pr}_{\sigma}$  is transitive.  $\square$

**Theorem 2** (Dissection of extended lattice gauge fields). *Every extended lattice gauge field  $\{\text{PT}_{\mathcal{C}}\}$  on  $(M, \mathcal{C})$  is equivalent to a standard lattice gauge field  $\text{PT}_{\Gamma}^0 : \mathcal{P}_{\Gamma}^0 \rightarrow G$ , together with a homotopy class of collections of glueing maps  $\{\{g_{\sigma\tau}\}_{\overline{c_{\sigma}} \subset \overline{c_{\tau}}}\}$  defined in (4.2), whose values at 0-cells are fixed and prescribed by  $\text{PT}_{\Gamma}^0$ . More explicitly, every extended lattice gauge field is equivalent to  $\text{PT}_{\Gamma}^0$  and a map which assigns, to every flag  $\overline{c_{\sigma}} \subset \overline{c_{\tau}}$  in  $\mathcal{C}$ ,  $c_{\sigma} \notin \mathcal{C}_0$ , the following collection of local homotopy data of glueing maps:*

(a) *To every 0-subcell  $\overline{c_{\sigma''}} \subseteq \overline{c_{\sigma}}$ , a group element*

$$g_{\sigma\tau}(p_{\sigma''}) = \text{PT}_{\Gamma}^0 \left( [\gamma_{\sigma''\sigma}]^{-1} \cdot [\gamma_{\sigma''\tau}] \right) \in G,$$

(b) *More generally, to every  $k$ -subcell  $\overline{c_{\sigma''}} \subseteq \overline{c_{\sigma}}$ ,  $k > 0$ , an extension class of maps  $[g_{\sigma\tau}|_{\overline{c_{\sigma''}}}]$  from  $\partial\overline{c_{\sigma''}}$  to  $\overline{c_{\sigma''}}$ , given the inductive boundary homotopy*

constraint in  $[S^{k-1}, G, g_{\partial\overline{c_{\sigma''}}}]$ , when  $k \geq 2$ ,<sup>6</sup>

$$(5.2) \quad [g_{\sigma\tau}|_{\partial\overline{c_{\sigma''}}}] = \left[ \sum_{\{\sigma''' \in \mathcal{C}_{k-1} \mid \sigma''' \supset \sigma''\}} g_{\sigma\tau}|_{\overline{c_{\sigma'''}}} \right],$$

which is regarded to be trivial when identified with the free homotopy class that contains it.

Moreover, the assignment is such that whenever we have that  $\sigma' \supset \sigma \supset \tau$ , the compatibility condition

$$g_{\sigma'\tau}(p_{\sigma''}) = g_{\sigma'\sigma}(p_{\sigma''}) \cdot g_{\sigma\tau}(p_{\sigma''}),$$

is satisfied at every 0-cell  $c_{\sigma''} \subset \overline{c_{\sigma'}}$ , and more generally, the compatibility condition

$$(5.3) \quad [g_{\sigma'\tau}|_{\overline{c_{\sigma''}}}] = [(g_{\sigma'\sigma}|_{\overline{c_{\sigma''}}}) \cdot (g_{\sigma\tau}|_{\overline{c_{\sigma''}}})],<sup>7</sup>$$

is satisfied at every  $k$ -cell  $c_{\sigma''} \subseteq \overline{c_{\sigma'}}$ ,  $k = 1, \dots, \dim(c_{\sigma'})$ .

*Proof.* Consider an arbitrary extended lattice gauge field  $\{\text{PT}_{\mathcal{C}}\}$ . Each class representative  $\text{PT}_{\mathcal{C}}$  induces a collection of glueing maps  $\{g_{\sigma\tau}\}$ , as the result of evaluating  $\text{PT}_{\mathcal{C}}$  at the family  $\mathcal{F}_{\sigma\tau}$ . All the values  $g_{\sigma\tau}(p_{\sigma''})$  at any 0-cell  $c_{\sigma''}$  for a given  $\overline{c_{\sigma}} \subset \overline{c_{\tau}}$  coincide, as they correspond to the evaluations of  $\text{PT}_{\Gamma}^0$  at the elements  $[\gamma_{\sigma''\sigma}^{-1}] \cdot [\gamma_{\sigma''\tau}]$  in the discrete groupoid  $\mathcal{P}_{\Gamma}^0$ . Hence, the homotopy class  $\{\text{PT}_{\mathcal{C}}\}$  is equivalent to a homotopy class of collections of glueing maps  $\{\{g_{\sigma\tau}\}_{\overline{c_{\sigma}} \subset \overline{c_{\tau}}}\}$  with fixed values over  $\mathcal{C}_0$ . The dissection of  $\{\text{PT}_{\mathcal{C}}\}$  as relative extension homotopy classes with fixed values over 0-cells is then straightforward. The restriction of the class representatives  $g_{\sigma\tau}$  to any subcell  $\overline{c_{\sigma''}}$  determines local homotopy classes of maps over  $\overline{c_{\sigma''}}$  relative to the 0-cells in its boundary, that can be factored as a necessarily trivial boundary homotopy class over  $\partial\overline{c_{\sigma''}}$ , together with an extension class to the interior  $c_{\sigma''}$ . Clearly, the classes associated to  $(k-1)$ -cells  $\overline{c_{\sigma'''}}$  in  $\partial\overline{c_{\sigma''}}$ , for any given  $k$ -cell  $c_{\sigma''}$ , glue according to lemma 4 to class of maps over  $\partial\overline{c_{\sigma''}}$  that are homotopic to a constant map. The homotopical data obtained this way clearly satisfies the compatibility conditions stated above.

The converse is verified in a similar way: as a consequence of lemma 4, given a collection of homotopy classes of extension maps to  $c_{\sigma''}$  for every flag  $\overline{c_{\sigma''}} \subset \overline{c_{\sigma}} \subset \overline{c_{\tau}}$ , with fixed values at 0-cells, and satisfying the glueing compatibility conditions above, it is possible to reconstruct a homotopy class of collections of glueing maps  $\{\{g_{\sigma\tau}\}_{\overline{c_{\sigma}} \subset \overline{c_{\tau}}}\}$ , relative to certain fixed values over their 0-cells, as there are no obstructions to extending a given map to the interior of a closed cell. What remains to be proved is that such relative

<sup>6</sup>The sum in the second term of the equality denotes the glueing of homotopy classes of cellularly smooth maps to  $S^{k-1}$  from lemma 4 for the map  $g_{\partial\overline{c_{\sigma''}}}$  of values at 0-cells.

<sup>7</sup>The right-hand side denotes the class of products of piecewise-smooth representatives in the relative extension classes  $[(g_{\sigma'\sigma}|_{\overline{c_{\sigma''}}})]$ ,  $[(g_{\sigma\tau}|_{\overline{c_{\sigma''}}})]$ , which clearly coincides with the corresponding relative extension class of any class representative (cf. lemma 5).

homotopy class of glueing maps is equivalent to an extended lattice gauge field  $\{\text{PT}_{\mathcal{E}}\}$ .

Let us consider any given representative collection of glueing maps  $\{g_{\sigma'\sigma}\}$ . With them, we will impose boundary conditions to construct a  $G$ -equivariant smooth horizontal distribution on the principal  $G$ -bundle  $P$  determined by the given clutching maps  $\{h_{vw}\}$ , by considering special  $G$ -equivariant lifts for choices of smooth collections of representatives of the elements in the complete path family  $\mathfrak{F}$ , satisfying a collection of compatibility conditions, that determine a collection of smooth distributions over the restrictions  $P|_{c_\sigma}$ . The constructed connection then determines an extended lattice gauge field for the starting collection of glueing maps. Let us then consider, for any flag  $c_{\sigma'} \subset \overline{c_\sigma}$ , a smooth family of paths  $\{\gamma^x\}_{x \in \overline{c_{\sigma'}}$  representing  $\mathcal{F}_{\sigma'\sigma} \in \mathfrak{F}$ . Consider any equivariant horizontal lifts of the families  $\{\gamma^x\}_{x \in \overline{c_{\sigma'}}$  satisfying the following conditions:

- (i) given any  $x \in \overline{c_{\sigma'}}$ , the parallel transport of the lifts of  $\gamma^x$  coincides with  $g_{\sigma'\sigma}(x)$ ;
- (ii) for any  $k$ -cell  $c_\sigma$ ,  $k \geq 1$ , and  $x \in \partial \overline{c_\sigma}$ , the collection of tangent lines<sup>8</sup> at  $x$  to all the paths  $\gamma^x$  in some family  $\mathcal{F}_{\sigma'\sigma}$  in  $\mathfrak{F}_\sigma$  span a vector space of dimension  $n - k + 1$ .

Then, it follows that for any  $x \in c_{\sigma'} \subset \partial \overline{c_\sigma}$ , the tangent lines at  $x$  for the corresponding paths in  $\mathfrak{F}_\sigma$ ,  $\mathfrak{F}_{\sigma'}$  are independent. In this way, the restriction  $P|_{c_\sigma}$ , for every  $k$ -cell  $c_\sigma$ , acquires a smooth  $(n - k + 1)$ -dimensional horizontal distribution, whose limit to  $\partial \overline{c_\sigma}$  is contained in the distributions over every subcell  $c_{\sigma'} \subset \partial \overline{c_\sigma}$ , and the corresponding limits for equidimensional neighboring cells coincide. There is no obstruction for the existence of such lifts. Moreover, there exists a smooth  $n$ -dimensional horizontal distribution on  $P$ , whose restriction to any cell  $c_\sigma$  contains the previous distribution on  $P|_{c_\sigma}$ . The latter horizontal distribution determines a smooth connection on  $P$  inducing the original glueing maps.  $\square$

Recall that every path subgroupoid  $\mathcal{P}_{\mathfrak{F}}$  has a special minimal subgroupoid  $\mathcal{P}_{\min}$  generated by the cellular path subfamilies  $\mathfrak{F}_{\min}$ , which consist of those paths generated by elements in  $\mathfrak{F}$ , whose source and target are base points of neighboring  $n$ -cells (remark 4). We can define larger homotopy classes of cellular parallel transport maps  $\{\text{PT}_{\mathcal{E}}\}_{\min}$ , with fixed values over the discrete subgroupoid  $\mathcal{P}_{\min}^0 = \mathcal{P}_{\mathfrak{F}}^0 \cap \mathcal{P}_{\min}$ , and consequently, a projection  $\pi_{\min}(\{\text{PT}_{\mathcal{E}}\})$  for every extended lattice gauge field onto the larger class that contains it.

**Definition 11.** The *core* of an extended lattice gauge field  $\{\text{PT}_{\mathcal{E}}\}$  is the homotopy class of cellular parallel transport maps with respect to  $\{\mathfrak{F}_{\min}\}$  containing  $\{\text{PT}_{\mathcal{E}}\}$ , that is,  $\{\text{PT}_{\mathcal{E}}\}_{\min} := \pi_{\min}(\{\text{PT}_{\mathcal{E}}\})$ .

<sup>8</sup>If the image of  $\gamma^x$  in  $M$  is not a manifold at  $x$ , we assume that  $\gamma^x$  is parametrized in such a way that one associates a pair of tangent lines, corresponding to the one-sided derivatives.



**Corollary 1.** *The core  $\{\text{PT}_{\mathcal{C}}\}_{\min}$  of an extended lattice gauge field  $\{\text{PT}_{\mathcal{C}}\}$  is equivalent to a collection of secondary local homotopy data, of the form:*

(a) *To every triple of elements  $c_{v_1}, c_{v_2}, c_{v_3} \in \mathcal{C}_n$  such that  $\overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} = \overline{c_{\sigma}}$  with  $c_{\sigma} \in \mathcal{C}_{n-2}$  as in (5.1), we assign*

- (i) *For every 0-cell  $c_{\sigma''} \subset \overline{c_{\sigma}}$ , a point  $\mathbf{h}_{\sigma}(c_{\sigma''}) \in V_G$ ,*
- (ii) *More generally, for every  $k$ -cell  $c_{\sigma'} \subseteq \overline{c_{\sigma}}$ ,  $k = 1, \dots, n-2$ , a relative homotopy class of maps*

$$[\mathbf{h}_{\sigma}|_{\overline{c_{\sigma'}}} : \overline{c_{\sigma'}} \rightarrow V_G],$$

*with fixed values over 0-subcells, representing an extension class from  $\partial\overline{c_{\sigma'}}$  to  $\overline{c_{\sigma'}}$ , that is determined by the inductive boundary data in  $[S^{k-1}, V_G, \mathbf{h}_{\partial\overline{c_{\sigma'}}}]$  when  $k \geq 2$ ,*

$$[\mathbf{h}_{\sigma}|_{\partial\overline{c_{\sigma'}}}] = \left[ \sum_{\{c_{\sigma''} \in \mathcal{C}_{k-1} : \sigma'' \supset \sigma'\}} \mathbf{h}_{\sigma}|_{\overline{c_{\sigma''}}} \right],^9$$

*which is regarded to be trivial when identified with the free homotopy class that contains it.*

*The assignment is equivariant for the permutation action of the group  $S_3$  on the triple  $c_{v_1}, c_{v_2}, c_{v_3}$  and its triadic action on  $V_G$ .*

(b) *To every pair of elements  $c_v, c_w \in \mathcal{C}_n$  such that  $\overline{c_v} \cap \overline{c_w} = \overline{c_{\tau}}$  with  $c_{\tau} \in \mathcal{C}_{n-1}$ , we assign an extension class  $[h_{vw} : \overline{c_{\tau}} \rightarrow G]$  of the inductively glued boundary class over  $\partial\overline{c_{\tau}}$  to  $\overline{c_{\tau}}$ , in such a way that the resulting induced glued class  $[h_{vw} \cdot h_{vw}]$  is homotopically trivial.*

*Proof.* Readily follows from the defining relation (4.4) of clutching maps and theorem 2, as there is an equivalence between the core of a lattice gauge field and a homotopy class of collections of clutching maps  $\{\{h_{vw}\}_{\overline{c_v} \cap \overline{c_w} \neq \emptyset}\}$  with fixed values over 0-cells. When the latter are dissected over compatible triples, we obtain the local homotopical data stated above, the first part corresponding to the data arising from any triple of cells  $\overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} = \overline{c_{\sigma}}$ , while the residual data takes the form of relative extension classes over all  $(n-1)$ -cells with orientation. Conversely, given any choice of such local homotopical data, it follows from the compatibility conditions, together with lemma 4, that the homotopy classes induced by the projections  $\pi_i(\mathbf{h}_{\sigma\sigma''})$  of the maps  $\mathbf{h}_{\sigma\sigma''}$  into components, with  $i = 1, 2, 3$ , together with the extension classes  $[h_{vw}]$ , determine a homotopy class of collections of clutching maps with fixed values over 0-cells.  $\square$

Given a triple  $(M, \mathcal{C}, \Gamma)$ , let us denote the space of standard lattice gauge fields for the cellular network  $\Gamma$  in  $M$  by  $\mathcal{M}_{\Gamma}$ . The space  $\mathcal{M}_{\Gamma}$  is isomorphic to the Lie group  $G^{N_1}$ , where  $N_1$  is the number of edges in the cellular network

<sup>9</sup>Here the sum denotes the glueing of homotopy classes of cellularly smooth maps constructed in lemma 4, while  $\mathbf{h}_{\partial\overline{c_{\sigma'}}}$  is the map of prescribed values at 0-cells. Following lemma 5, the set of such extension classes is a torsor for  $\pi_k(V_G, \mathbf{e})$ , where  $\mathbf{e} = (e, e, e)$  (cf. corollary 2).



$\Gamma$ , and consequently inherits a smooth manifold structure. Moreover, let us denote the corresponding space of extended lattice gauge fields by  $\mathcal{M}_{\mathcal{E}}$ . There is an obvious projection

$$(5.4) \quad \text{pr}_{\mathcal{E}} : \mathcal{M}_{\mathcal{E}} \rightarrow \mathcal{M}_{\Gamma}.$$

In a similar way, if we let  $\mathcal{N}_{\mathcal{E}}$  be the space of cores of extended lattice gauge field, and  $\text{pr}_{\Gamma, \min} : \mathcal{M}_{\Gamma} \rightarrow \mathcal{N}_{\min}$  the projection to core network data, there is also an additional pair of projections

$$(5.5) \quad \text{pr}_{\min} : \mathcal{N}_{\mathcal{E}} \rightarrow \mathcal{N}_{\min},$$

and  $\text{pr}_{\text{core}} : \mathcal{M}_{\mathcal{E}} \rightarrow \mathcal{N}_{\mathcal{E}}$ , in such a way that we have the following commutative diagram

$$(5.6) \quad \begin{array}{ccc} \mathcal{M}_{\mathcal{E}} & \xrightarrow{\text{pr}_{\text{core}}} & \mathcal{N}_{\mathcal{E}} \\ \text{pr}_{\mathcal{E}} \downarrow & & \downarrow \text{pr}_{\min} \\ \mathcal{M}_{\Gamma} & \xrightarrow{\text{pr}_{\Gamma, \min}} & \mathcal{N}_{\min} \end{array}$$

It turns out that the projections (5.4) and (5.5) are covering maps. We now describe the structure of these covering maps in terms of “generators and relations” for their groups of deck transformations. Consider the group

$$\tilde{G}_{\mathcal{E}} = \prod_{\overline{c_{\sigma}} \subset \overline{c_{\tau}}} \left( \prod_{\overline{c_{\sigma'}} \subseteq \overline{c_{\sigma}}} G_{\sigma'} \right),$$

where  $G_{\sigma'} = \pi_k(G, e)$  for  $c_{\sigma'} \in \mathcal{C}_k$  (in particular,  $G_{\sigma'} = \{e\}$  when  $k = 0$ ). We will consider a subgroup determined by a series of group relations of two kinds. For any flag  $\overline{c_{\sigma'}} \subset \overline{c_{\sigma}} \subset \overline{c_{\tau}}$ , and  $\overline{c_{\sigma''}} \subseteq \overline{c_{\sigma'}}$ , consider the homomorphisms

$$\Theta_{\sigma' \sigma \tau}^{\sigma''} : \tilde{G}_{\mathcal{E}} \times \tilde{G}_{\mathcal{E}} \times \tilde{G}_{\mathcal{E}} \rightarrow G_{\sigma''}$$

$$\Theta_{\sigma' \sigma \tau}^{\sigma''} = (\text{pr}_{\sigma'' \sigma' \sigma}) (\text{pr}_{\sigma'' \sigma \tau}) (\text{pr}_{\sigma'' \sigma' \tau})^{-1}.$$

The kernels of these maps among homotopy groups give relations that we will call *multiplicative relations in homotopy*. The second type of relations are described as follows. For any  $(k+1)$ -cell  $c_{\sigma}$ ,  $k = 1, \dots, n-2$ , define the product homomorphism

$$\alpha_{\sigma} : \prod_{c_{\sigma'}^k \subset \partial \overline{c_{\sigma}}} G_{\sigma'} \rightarrow \pi_k(G, e)$$

which is unambiguous since  $\pi_k(G, e)$  is abelian. Its kernel determines a relation that we will call *boundary relation in homotopy*. On the other

hand, consider the group

$$\tilde{H}_{\mathcal{C}} = \left( \prod_{\substack{c_u, c_v, c_w \in \mathcal{C}_n \\ \overline{c_u} \cap \overline{c_v} \cap \overline{c_w} = \overline{c_\sigma}}} \left( \prod_{\overline{c_{\sigma'}} \subseteq \overline{c_\sigma}} H_{\sigma'} \right) \right) \times \left( \prod_{\substack{c_v, c_w \in \mathcal{C}_n \\ \overline{c_v} \cap \overline{c_w} \neq \emptyset}} H_{vw} \right)$$

where  $H_{\sigma'} = \pi_k(V_G, \mathbf{e})$  if  $c_{\sigma'} \in \mathcal{C}_k$ , and  $H_{vw} = \pi_{n-1}(G, e)$ . Consider the homomorphisms

$$\begin{aligned} \mu_{vw} : H_{vw} \times H_{vw} &\rightarrow \pi_{n-1}(G, e) \\ ([g], [g']) &\mapsto [gg'] \end{aligned}$$

which, by a slight abuse of notation, will be regarded as defined on  $\tilde{H}_{\mathcal{C}} \times \tilde{H}_{\mathcal{C}}$  after composing with a suitable projection map. In particular, we have that  $\ker(\mu_{vw}) = \ker(\mu_{wv})$ . Similarly, for  $c_\sigma \in \mathcal{C}_{k+1}$ ,  $k = 1, \dots, n-3$  define the product homomorphism

$$\beta_\sigma : \prod_{c_{\sigma'}^k \subset \partial \overline{c_\sigma}} \pi_k(V_G, \mathbf{e}) \rightarrow \pi_k(V_G, \mathbf{e})$$

whose kernel defines an analogous boundary homotopy relation. Finally, we can consider a triadic action of  $S_3$  in  $\tilde{H}_{\mathcal{C}}$ , by acting in the obvious way in the first component, and as the identity in the second component. Such action determines an invariant subgroup  $\tilde{H}_{\mathcal{C}}^{S_3} \subset \tilde{H}_{\mathcal{C}}$ .

**Corollary 2.** *The projections (5.4) and (5.5) are covering maps. More concretely, for any given standard lattice gauge field  $\text{PT}_\Gamma^0 : \mathcal{P}_\Gamma^0 \rightarrow G$  in  $\mathcal{M}_\Gamma$ , the set of extended lattice gauge fields in  $\text{pr}_{\mathcal{C}}^{-1}(\text{PT}_\Gamma^0)$  is a torsor for the group*

$$G_{\mathcal{C}} = \left( \bigcap_{\overline{c_{\sigma'}} \subseteq \overline{c_\sigma} \subset \overline{c_\tau}} \ker(\alpha_{\sigma'}) \right) \cap \left( \bigcap_{\overline{c_{\sigma''}} \subseteq \overline{c_{\sigma'}} \subset \overline{c_\sigma} \subset \overline{c_\tau}} \ker(\Theta_{\sigma'\sigma\tau}^{\sigma''}) \right)$$

*Similarly, the set of cores of extended lattice gauge fields in  $\text{pr}_{\min}^{-1}(\text{PT}_{\min}^0)$  is a torsor for the subgroup*

$$H_{\min} = \left( \bigcap_{\substack{\overline{c_{\sigma'}} \subseteq \overline{c_\sigma} = \overline{c_u} \cap \overline{c_v} \cap \overline{c_w} \\ c_u, c_v, c_w \in \mathcal{C}_n}} \ker(\beta_{\sigma'}) \right) \cap \left( \bigcap_{\substack{c_v, c_w \in \mathcal{C}_n \\ \overline{c_v} \cap \overline{c_w} \neq \emptyset}} \ker(\mu_{vw}) \right) \cap \tilde{H}_{\mathcal{C}}^{S_3}$$

*Proof.* The covering map property follows from the fact that there is always a homotopy isomorphism between  $\text{pr}_{\mathcal{C}}^{-1}(\text{PT}_\Gamma^0)$  (resp.  $\text{pr}_{\min}^{-1}(\text{PT}_{\min}^0)$ ), for any  $\text{PT}_\Gamma^0 \in \mathcal{M}_\Gamma$  (resp.  $\text{PT}_{\min}^0 \in \mathcal{N}_{\min}$ ) and the inverse image of any sufficiently small deformation of it.

The fact that the group  $G_{\mathcal{C}}$  acts on the set of extended lattice gauge fields compatible with a given standard lattice gauge field  $\text{PT}_\Gamma^0 \in \mathcal{M}_\Gamma$  is essentially

a consequence of lemma 5: the action of a given component on an extension class is given by pointwise left multiplication of representatives, the boundary relations in homotopy are introduced to ensure that the vanishing of the obstruction to the existence of extensions to the interior of a cell  $\partial\overline{c_{\sigma'}}$  on an extended lattice gauge field is preserved under the action on  $G_{\mathcal{C}}$ . Similarly, the multiplicative relations in homotopy are introduced to ensure that the compatibility conditions (5.3) of an extended lattice gauge field are preserved under the action on  $G_{\mathcal{C}}$ , a fact that is possible due to the abelian nature of the homotopy groups of  $G$  and that the action is left and right simultaneously. Moreover, since for a given  $k$ -cell  $c_{\sigma'}$ , the set of extension classes to its interior, given a fixed boundary homotopy data is a torsor for  $\pi_k(G, e)$ , it readily follows that the stabilizer in  $G_{\mathcal{C}}$  of any extended lattice gauge field is equal to the identity.

The same applies for the group  $H_{\min}$  acting on the set of cores compatible with a given groupoid homomorphism  $\text{PT}_{\min}^0 : \mathcal{P}_{\min}^0 \rightarrow G$ , recalling the defining relation (4.4), as a consequence of corollary 1. The invariance of  $H_{\min}$  under the triadic action ensures that the action of any of its elements in a core is also invariant under the corresponding triadic action. Moreover, as before, the boundary relations in homotopy ensure that the vanishing of obstructions to existence of extensions are preserved under the action of  $H_{\min}$  on a core. Together with the inversion relations determined by all  $\ker(\mu_{vw})$ , we ensure that conditions (a) and (b) in corollary 1 are preserved.  $\square$

*Remark 13.* The dissection of the core of an extended lattice gauge field in corollary 1, or what is the same, the relative homotopy classes of clutching maps, is strictly dependent on the fixed values over 0-cells of the latter, that is, the a priori choice of a collection of reference points

$$\bigsqcup_{c_{\sigma} \in \mathcal{C}_{n-2}} \left\{ \mathbf{h}_{\sigma} \left( c_{\sigma_0''} \right) : c_{\sigma_0''} \in \mathcal{C}_0, \sigma_0'' \supset \sigma \right\}.$$

Such is the basis for the contrast between an intrinsically *local* object (the extended lattice gauge fields), and the *global* notion of cellular bundle data representing an equivalence class of principal  $G$ -bundles: if a different choice of reference points is made, there would still exist a corresponding relative homotopy class of clutching maps defining an equivalent principal  $G$ -bundle, but the new extension classes resulting from the dissection procedure may turn out to be completely different than the previous ones. The interest in getting a better understanding of the correspondence between the extended lattice gauge fields and the equivalence classes of principal  $G$ -bundles motivates an induced notion of cellular equivalence of extended lattice gauge fields, following from definition 9. Namely, two extended lattice gauge fields  $\{\text{PT}_{\mathcal{C}}\}, \{\text{PT}'_{\mathcal{C}}\}$  are said to be *cellularly equivalent* if the principal  $G$ -bundles they determine are equivalent. The projections (5.4) and (5.5) provide an alternative characterization of such cellular equivalence, since by definition,

the core of an extended lattice gauge field is the minimal local homotopy data (relative to  $\mathcal{C}$ ) extending any given standard lattice gauge field that is necessary to reconstruct a principal  $G$ -bundle  $P \rightarrow M$ , up to equivalence. Such characterization is described in the following result.

**Proposition 2.** *Two extended lattice gauge fields  $\{\text{PT}_{\mathcal{C}}\}$  and  $\{\text{PT}'_{\mathcal{C}}\}$  in  $\mathcal{M}_{\mathcal{C}}$  yield equivalent principal  $G$ -bundles if and only if they lie in the same connected component in  $\mathcal{M}_{\mathcal{C}}$ .*

*Proof.* The covering map (5.5) was defined to correspond to the cellular equivalence in definition 9, as the connected components of  $\mathcal{N}_{\mathcal{C}}$  are in natural correspondence with the equivalence classes of principal  $G$ -bundles. Then, let us assume that two extended lattice gauge fields project to the same connected component in  $\mathcal{N}_{\mathcal{C}}$ , i.e. any pair of representatives of their cores are cellularly equivalent. Using an isomorphism, we may assume that the bundles they determine are the same, equal to  $P$ . Consider any pair of gauge classes of smooth connections representing  $\{\text{PT}_{\mathcal{C}}\}$  and  $\{\text{PT}'_{\mathcal{C}}\}$ . Since the space  $\mathcal{A}_P/\mathcal{G}_P$  is connected, there is a path connecting both gauge classes of connections. Such path determines a path in  $\mathcal{M}_{\mathcal{C}}$  connecting  $\{\text{PT}_{\mathcal{C}}\}$  and  $\{\text{PT}'_{\mathcal{C}}\}$ , and therefore, the inverse image under  $\text{pr}_{\text{core}}$  of a connected component in  $\mathcal{N}_{\mathcal{C}}$  is a connected component in  $\mathcal{M}_{\mathcal{C}}$ . This determines a bijective correspondence between the connected components of  $\mathcal{M}_{\mathcal{C}}$  and  $\mathcal{N}_{\mathcal{C}}$ , and implies the result.  $\square$

**Corollary 3.** *Cellular equivalence determines fibrations*

$$\Phi : \mathcal{M}_{\mathcal{C}} \rightarrow \check{H}^1(M, \underline{G}), \quad \Psi : \mathcal{N}_{\mathcal{C}} \rightarrow \check{H}^1(M, \underline{G})$$

such that  $\Phi = \Psi \circ \text{pr}_{\text{core}}$ , given by mapping the connected components in  $\mathcal{M}_{\mathcal{C}}$  (resp.  $\mathcal{N}_{\mathcal{C}}$ ) to their corresponding isomorphism classes of principal  $G$ -bundles. In particular, there exists subgroups  $K_{\mathcal{C}} \subset G_{\mathcal{C}}$  and  $K_{\min} \subset H_{\min}$  such that

$$(5.7) \quad G_{\mathcal{C}}/K_{\mathcal{C}} \cong H_{\min}/K_{\min} \cong \check{H}^1(M, \underline{G}),$$

that are the subgroups of deck transformations in  $\mathcal{M}_{\mathcal{C}}$  (resp.  $\mathcal{N}_{\mathcal{C}}$ ) preserving any given connected component, and thus stabilizing any given equivalence class of principal  $G$ -bundles  $\{P\}$ .

Having understood the way a discretization of a gauge field leads to the recovery of the topology of a principal  $G$ -bundle, we can now proceed and describe the global geometric picture that gives rise to the space of extended lattice gauge fields  $\mathcal{M}_{\mathcal{C}}$ . In general, we have the decomposition of  $\mathcal{M}_{\mathcal{C}}$  into connected components

$$\mathcal{M}_{\mathcal{C}} = \bigsqcup_{\{P\} \in \check{H}^1(M, \underline{G})} \mathcal{M}_{P, \mathcal{C}}$$

Let us consider any given principal  $G$ -bundle  $P$ , with associated space of smooth connections modulo gauge transformations  $\mathcal{A}_P/\mathcal{G}_P$ . We have seen that the gauge group  $\mathcal{G}_P$  contains a normal subgroup  $\mathcal{G}_{P,*}$ , that we are calling

the *restricted gauge group* (cf. [3]), consisting of those gauge transformations whose value over the fibers of every base point in  $(M, \mathcal{C})$  is the identity, with the characteristic feature of stabilizing any given smooth cellular parallel transport map  $\text{PT}_{\mathcal{C}}$ . Therefore, there is a bijective correspondence between the space of smooth cellular parallel transport maps on  $(M, \mathcal{C})$  yielding a bundle isomorphic to  $P$ , and the quotient  $\mathcal{A}_P/\mathcal{G}_{P,*}$ . Moreover, there is a principal bundle

$$(5.8) \quad \mathcal{A}_P/\mathcal{G}_{P,*} \rightarrow \mathcal{A}_P/\mathcal{G}_P,$$

whose structure group is the finite-dimensional quotient Lie group  $\mathcal{G}_{P,\mathcal{C}} = \mathcal{G}_P/\mathcal{G}_{P,*}$ .

**Definition 12.** For  $\mathcal{E}_{\mathcal{C}}$  a set of fiber points in  $P$  covering a set of based points  $\mathcal{B}_{\mathcal{C}}$  in  $(M, \mathcal{C})$ , we say that two smooth connections over  $(P, \mathcal{E}_{\mathcal{C}})$  are  $\mathcal{C}$ -*equivalent*, or *equivalent at scale  $\mathcal{C}$* , if their associated cellular parallel transport maps project to the same point in  $\mathcal{M}_{P,\mathcal{C}}$ . By a *microscopical deformation* of  $A \in \mathcal{A}_P$ , relative to a cellular network in  $(M, \mathcal{C})$ , we mean a smooth path in  $\mathcal{A}_P$  preserving  $\mathcal{C}$ -equivalence and starting at  $A$ .

A fundamental by-product of theorem 1 and corollary 1 is the *cellular homotopy fibration*

$$\mathcal{A}_P/\mathcal{G}_{P,*} \rightarrow \mathcal{M}_{P,\mathcal{C}}, \quad \text{PT}_{\mathcal{C}} \mapsto \{\text{PT}_{\mathcal{C}}\}$$

mapping a given cellular parallel transport map to the extended lattice gauge field it induces. In this way, the essence of definition 12 is captured in a commutative diagram of principal  $\mathcal{G}_{P,\mathcal{C}}$ -bundles

$$(5.9) \quad \begin{array}{ccc} \mathcal{A}_P/\mathcal{G}_{P,*} & \longrightarrow & \mathcal{M}_{P,\mathcal{C}} \\ \downarrow & & \downarrow \\ \mathcal{A}_P/\mathcal{G}_P & \longrightarrow & \mathcal{M}_{P,\mathcal{C}}/\mathcal{G}_{P,\mathcal{C}} \end{array}$$

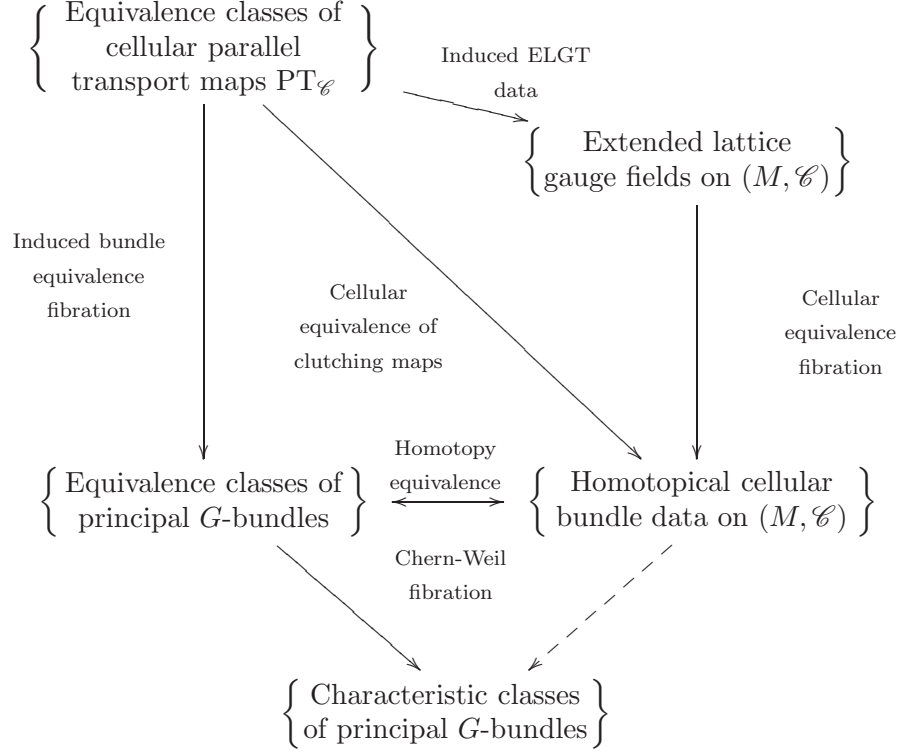
Consequently, the spaces of geometric significance in the proposed extended lattice gauge theory are the quotient spaces

$$\mathcal{M}_{P,\mathcal{C}}/\mathcal{G}_{P,\mathcal{C}},$$

which play the role of *finite dimensional analogs* of the spaces  $\mathcal{A}_P/\mathcal{G}_P$ , for every  $\{P\} \in \check{H}^1(M, \underline{G})$ .

A fundamental consequence of corollaries 2 and 3 is the reconstruction of the space  $\check{H}^1(M, \underline{G})$  of isomorphism classes of principal  $G$ -bundles as a homogeneous space (5.7), in terms of homotopy data of a local nature. In such sense, understanding the space  $\check{H}^1(M, \underline{G})$  has been reduced to the understanding of the groups  $G_{\mathcal{C}}$  and  $K_{\mathcal{C}}$ . We consider that such a hybrid (geometric/algebraic) structure deserves to be thoroughly studied, as it sheds new light into the spaces  $\check{H}^1(M, \underline{G})$  on arbitrary manifolds, but the task seems to be far from trivial. Therefore, we plan to return to it in a subsequent publication.

Altogether, we have constructed a series of fibrations that take the set of cellular parallel transport maps over  $(M, \mathcal{C})$  as a starting point. Such fibrations could be understood diagrammatically as follows:<sup>10</sup>



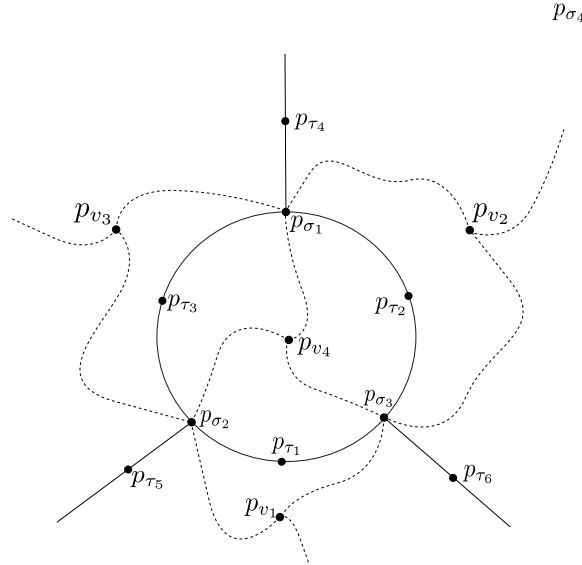
## 6. EXTENDED LATTICE GAUGE FIELDS IN SMALL DIMENSIONS

To supplement section 5, we provide explicit homotopical characterizations of extended lattice gauge fields following from theorem 2 and corollary 1, for oriented manifolds of dimensions 2, 3 and 4. In fact, such characterizations suggest a recursive algorithm that could be implemented in the general case to determine a set of generators for the extension classes, which is based on a systematic use of the compatibility conditions (5.3). It is important to keep in mind that for any Lie group  $G$ , the group  $\pi_1(G, e)$  is abelian,  $\pi_2(G, e)$  is always trivial, while  $\pi_3(G, e)$  is always torsion-free, and hence isomorphic to  $\mathbb{Z}^m$  for some  $m$  (see [13]).

<sup>10</sup>So far, we have excluded the final topological fibration to the characteristic classes of a principal  $G$ -bundle in the present work. We plan to study the realization of characteristic classes from homotopical cellular bundle data, in the sense of the Chern-Weil theory, in a separate publication, as such problem is fundamental for its own sake.

• ( $n = 2$ ; oriented surfaces) This is the simplest nontrivial case. The triangle-dual cellular decompositions are those for which at each vertex, exactly 3 edges merge (5.1). Particular examples are the tetrahedral, cubical and dodecahedral cellular representations of the 2-sphere. In such case, the only flags one needs to consider take the form  $\overline{c_\tau} \subset \overline{c_v}$  with  $c_\tau \in \mathcal{C}_1$ , and an extended lattice gauge field is simply a sort of splitting of its core: the latter corresponds to (i) an assignment of a parallel transport to every path of the form  $[\gamma_{\sigma v}]^{-1} \cdot [\gamma_{\sigma w}]$  for any pair of 2-cells  $\overline{c_v}, \overline{c_w}$  sharing a common boundary  $\overline{c_\tau}$  containing a 0-cell  $c_\sigma$ , and (ii) a collection of extension classes  $[h_{vw}]$  relative to their boundary values, which group together as the points  $\{\mathbf{h}_\sigma\} \subset V_G$ . In turn, an extended lattice gauge field corresponds to (i) an assignment of a parallel transport to every path of the form  $[\gamma_{\sigma\tau}]^{-1} \cdot [\gamma_{\sigma v}]$ ,<sup>11</sup> and (ii) a collection of extension classes  $[g_{\tau v}]$  relative to their fixed (and compatible) boundary values  $g_{\tau v}(p_\sigma), g_{\tau v}(p_{\sigma'}), c_\sigma, c_{\sigma'} \subset \partial \overline{c_\tau}$ . Hence, every class representative  $h_{vw}$  splits as  $g_{\tau v}^{-1} \cdot g_{\tau w}$ , for a pair of class representatives  $g_{\tau v}, g_{\tau w}$ . All of such extension classes can be parametrized by elements in  $\pi_1(G, e)$  once auxiliary choices of extension maps are made. The boundary homotopy constraints (5.2) and the compatibility conditions (5.3) are vacuous in this case.

FIGURE 3. Stereographic projection of a tetrahedral cellular decomposition of the 2-sphere, together with a cellular network. Base points are indicated with the letter  $p$ . The 2-cells are labelled with the subscripts  $v_i$ , the 1-cells with  $\tau_j$ , and the 0-cells with  $\sigma_k$ .



<sup>11</sup>Observe that  $[\gamma_{\sigma v}]^{-1} \cdot [\gamma_{\sigma w}] = ([\gamma_{\sigma\tau}]^{-1} \cdot [\gamma_{\sigma v}])^{-1} \cdot ([\gamma_{\sigma\tau}]^{-1} \cdot [\gamma_{\sigma w}])$ .

In fact, more can be said in dimension 2, regarding the projection of an extended lattice gauge field to its cellular bundle data. We can prescribe an equivalence class of principal  $G$ -bundles by means of a “canonical form”, under cellular equivalence, of an extended lattice gauge field. Recall that the equivalence classes of principal  $G$ -bundles over an oriented surface  $S$  are parametrized by  $\pi_1(G, e)$ , and correspond to homotopy classes of transition functions for a trivialization  $\{S \setminus \{p\}, \mathcal{U}\}$ , where  $p$  is an arbitrary point in  $S$  and  $\mathcal{U}$  is a small disk containing  $p$ , after a retraction from  $(S \setminus \{p\}) \cap \mathcal{U}$  to a choice of some simply closed loop  $\gamma \subset (S \setminus \{p\}) \cap \mathcal{U}$  is made. If we let  $\gamma$  be the boundary of a 2-cell  $c_v \in \mathcal{C}$ ,  $p = p_v$ , and  $\mathcal{U} = \mathcal{U}_v$ , we can recover all equivalence classes by choosing any 1-cell  $c_\tau \subset \partial \overline{c_v}$  (such that  $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$ ), choosing arbitrary elements in  $V_G$  for every  $c_\sigma \in \mathcal{C}_0$ , and decreeing all extension classes of clutching maps to be trivial, except for  $[h_{vw}]$ . The correspondence of such data and the equivalence classes of principal  $G$ -bundles then follows when the extension classes of clutching maps over the 1-subcells in  $\partial \overline{c_v}$  are glued.

- ( $n = 3$ ) Over an oriented 3-manifold, besides the prescription of a standard lattice gauge field, with generators corresponding to the parallel transports of all paths  $[\gamma_{\sigma'_0 \sigma}]$  joining the base point of a  $k$ -cell  $c_\sigma$ ,  $k = 1, 2, 3$ , and a 0-cell  $c_{\sigma'_0}$  in its boundary, the compatible values at 0-cells  $c_{\sigma'_0}$  for the relative homotopy classes of glueing maps can be constructed as

$$g_{\sigma\tau}(p_{\sigma'_0}) = \text{PT}^0 \left( [\gamma_{\sigma'_0 \sigma}] \cdot [\gamma_{\sigma'_0 \tau}] \right).$$

Then, an extended lattice gauge field can be entirely described in terms of the flags of the form  $\overline{c_\sigma} \subset \overline{c_\tau}$  and  $\overline{c_\sigma} \subset \overline{c_v}$ , with  $c_\sigma \in \mathcal{C}_1$ ,  $c_\tau \in \mathcal{C}_2$ ,  $c_v \in \mathcal{C}_3$ . This is so since the relative homotopy classes  $[g_{\sigma\tau}]$ ,  $[g_{\sigma v}]$  are determined by a single extension to  $\overline{c_\sigma}$ , which can be parametrized by elements in the group  $\pi_1(G, e)$  after an auxiliary choice of extension maps is made. Then, the extensions of a class  $[g_{\tau v}]$  over the boundary 1-subcells  $\overline{c_\sigma}$  are determined recursively from the factorization property of representatives following (5.3)

$$g_{\tau v}|_{\overline{c_\sigma}} = g_{\sigma\tau}^{-1} \cdot g_{\sigma v},$$

and must satisfy that their glueing over  $\partial \overline{c_\tau}$  is homotopic to a point in  $G$ , according to (5.2). The remaining extension class for  $[g_{\tau v}]$  to the 2-cell  $c_\tau$  is necessarily trivial, as a consequence of the triviality of  $\pi_2(G, e)$ .

- ( $n = 4$ ) This case is also relatively easy to describe. Let  $c_{\sigma'} \in \mathcal{C}_1$ ,  $c_\sigma \in \mathcal{C}_2$ ,  $c_\tau \in \mathcal{C}_3$ ,  $c_v \in \mathcal{C}_4$ . Once a standard lattice gauge field has been prescribed in terms of a set generators as in the case  $n = 3$ , and the induced choice of all compatible values at 0-cells for the relative homotopy classes of glueing maps is constructed, we proceed as follows. To construct the extension classes, the fundamental scaffolding is determined by the relative homotopy classes  $[g_{\sigma'\sigma}]$ ,  $[g_{\sigma'\tau}]$  and  $[g_{\sigma'v}]$ , which are determined by an extension class from  $\partial \overline{c_{\sigma'}}$  to  $\overline{c_{\sigma'}}$ . As before, such extension classes can be parametrized by elements in  $\pi_1(G, e)$ , once an auxiliary choice of extension map is made. The



homotopy classes  $[g_{\sigma\tau}|_{\overline{c_{\sigma'}}}]$  and  $[g_{\tau v}|_{\overline{c_{\sigma'}}}]$  are determined once again from the factorization property of representatives

$$g_{\sigma\tau}|_{\overline{c_{\sigma'}}} = g_{\sigma'\sigma}^{-1} \cdot g_{\sigma'\tau}, \quad g_{\tau v}|_{\overline{c_{\sigma'}}} = g_{\sigma'\tau}^{-1} \cdot g_{\sigma'v},$$

which, in turn, determine the classes  $[g_{\sigma v}|_{\overline{c_{\sigma'}}}]$ . Furthermore, once the boundary homotopy constraint (5.2) is imposed on the glued classes  $[g_{\sigma\tau}|_{\partial\overline{c_{\sigma}}}]$  and  $[g_{\tau v}|_{\partial\overline{c_{\sigma}}}]$  (which, in turn, imply the same constraint for the class  $[g_{\sigma v}|_{\partial\overline{c_{\sigma}}}]$ ), their extensions from  $\partial\overline{c_{\sigma}}$  to  $\overline{c_{\sigma}}$  are trivial due to the triviality of  $\pi_2(G, e)$ . For the same reason, the boundary homotopy constraint for  $[g_{\tau v}|_{\partial\overline{c_{\tau}}}]$  is automatically verified. To conclude, we only need to prescribe the extension of  $[g_{\tau v}]$  from  $\partial\overline{c_{\tau}}$  to  $\overline{c_{\tau}}$ , which can be parametrized by elements in  $\pi_3(G, e)$  once an auxiliary choice of extension map is made.

A minimal example that holds in the previous cases, and in fact, for arbitrary dimensions, is the  $n$ -sphere, with a cellular decomposition induced from thinking about it as the boundary of an  $(n+1)$ -simplex or an  $(n+1)$ -cube. For instance, when  $n = 2$ , and  $S^2$  corresponds to the boundary of a tetrahedron (figure 3), we can list a pair of paths for every 1-cell  $c_{\tau}$ , joining  $p_{\tau}$  and the 0-cells in its boundary  $c_{\sigma_1}, c_{\sigma_2}$ . Each of these paths gets assigned a corresponding parallel transport  $\text{PT}^0([\gamma_{\sigma_i\tau}])$ , playing the role of a prototypical lattice gauge field, in the usual sense. Now, the previous paths get complemented with the paths  $[g_{\sigma'_0\sigma}]$  in a cellular network, joining the base point of an arbitrary  $k$ -cell  $c_{\sigma}$ ,  $k \geq 2$ , to a 0-cell  $c_{\sigma'_0}$  in its boundary, also getting assigned a parallel transport, and together fully determining a standard lattice gauge field. Finally, the prescription gets completed with an extension class of maps for every flag  $\overline{c_{\sigma}} \subset \overline{c_{\tau}}$ , and every  $k$ -subcell  $\overline{c_{\sigma'}} \subseteq \overline{c_{\sigma}}$ . When  $d = 2$ , there is one such extension class for every flag  $\overline{c_{\tau}} \subset \overline{c_v}$  with  $c_{\tau} \in \mathcal{C}_1$ , and the extensions for the pairs  $\overline{c_{\tau}} \subset \overline{c_v}$  and  $\overline{c_{\tau}} \subset \overline{c_w}$ , such that  $\overline{c_{\tau}} = \overline{c_v} \cap \overline{c_w}$ , are inverse to each other.

## 7. PACHNER MOVES AND INDEPENDENCE OF CELLULAR DECOMPOSITIONS

So far nothing has been said about the potential independence of our geometric and topological constructions under changes of the underlying triangle-dual cellular decomposition  $\mathcal{C}$  that is required to be chosen. A rather special feature of the category of triangle-dual cellular decompositions on a manifold  $M$ , due to the simplicial nature of its objects, is the abundance of morphisms that allows us to connect and compare any given pair of them, up to dual P.L. equivalence, in a systematic way. Namely, it was proved by Pachner [16] that any two smooth triangulations of a manifold are related by a sequence of the so-called *Pachner moves*. Let  $X^{n+1}$  be an abstract  $(n+1)$ -simplex. By definition, we say that two different P.L. structures  $\Delta : |K| \rightarrow M$  and  $\Delta' : |K'| \rightarrow M$  on a manifold  $M$  differ by a Pachner move if there exists a pair of injective simplicial maps  $\mu : L \rightarrow K$  and  $\mu' : L' \rightarrow K'$ ,

where

$$L = \bigcup_{l=0}^k X_l^n \subset \partial X^{n+1}, \quad L' = \bigcup_{l=k+1}^{n+1} X_l^n \subset \partial X^{n+1},$$

for some arbitrary labeling  $X_0^n, \dots, X_{n+1}^n$  of the  $n$ -simplices in  $\partial X^{n+1}$  and  $0 \leq k \leq n$ , such that

- (i) the map  $(\Delta')^{-1} \circ \Delta : |K| \rightarrow |K'|$  is simplicial outside  $|\mu(L \setminus \partial L)|$ ,
- (ii)  $\Delta(|\mu(L)|) = \Delta'(|\mu'(L')|)$ ,
- (iii)  $\Delta(|\mu(\sigma)|) = \Delta'(|\mu'(\sigma)|)$  for all  $\sigma \in \partial L = \partial L'$ .

Consequently, we are left to conclude that any two triangle-dual cellular decompositions  $\mathcal{C}$  and  $\mathcal{C}'$  of an  $n$ -manifold  $M$  are related by a corresponding sequence of dual Pachner moves – the corresponding transformations over the cell decompositions. Our first step towards cellular independence is to provide an elementary description of such transformations.

*Remark 14.* In the previous definition of a Pachner move, the  $k+1$  different  $n$ -simplices of the simplicial complex  $L$  meet at a common  $(n-k)$ -face  $\sigma$ , while the  $n-k+1$  different  $n$ -simplices of the simplicial complex  $L'$  meet at a common  $k$ -face  $\sigma'$ , dual to  $\sigma$  (in the case when  $k=0$ ,  $L=\sigma$  and  $\sigma'$  is its complementary vertex  $v$  in  $X^{n+1}$ , and correspondingly for  $k=n$ ). Therefore, when two triangulations  $\Delta$  and  $\Delta'$  of  $M$  differ by a Pachner move, such operation may be interpreted as the replacement of the  $(n-k)$ -face  $\sigma$  by the  $k$ -face  $\sigma'$  (unless  $k=0$ , where the Pachner move consists of a refinement of an  $n$ -simplex through the extra interior vertex  $v$ , and conversely for  $k=n$ ), and there is a common refinement  $\Delta''$  containing exactly an additional 0-simplex  $v$ , corresponding to the intersection of  $\sigma$  and  $\sigma'$ . In turn,  $\mathcal{C} \setminus \{c_\sigma\} \cong \mathcal{C}' \setminus \{c_{\sigma'}\}$ , and the corresponding triangle-dual cellular decomposition  $\mathcal{C}''$  contains an additional  $n$ -cell  $c_v$  than  $\mathcal{C}$  and  $\mathcal{C}'$  (figure 4). The  $n$ -cell closure  $\overline{c_v}$  is topologically equivalent to the product  $\overline{c_\sigma} \times \overline{c_{\sigma'}}$ .

**Lemma 6.** *Any two triangle-dual cellular decompositions  $\mathcal{C}$  and  $\mathcal{C}'$  of  $M$ , arising from triangulations  $\Delta$  and  $\Delta'$ , and related by a Pachner move as before, are degenerations of a smooth 1-parameter family of triangle-dual cellular decompositions  $\{\mathcal{C}_t''\}_{t \in (0,1)}$ ,*

$$\lim_{t \rightarrow 0^+} \mathcal{C}_t'' = \mathcal{C}, \quad \lim_{t \rightarrow 1^-} \mathcal{C}_t'' = \mathcal{C}',$$

with  $\mathcal{C}_t''$  dual to  $\Delta''$ , where  $\Delta''$  is the common refinement of  $\Delta$  and  $\Delta'$  described in remark 14.

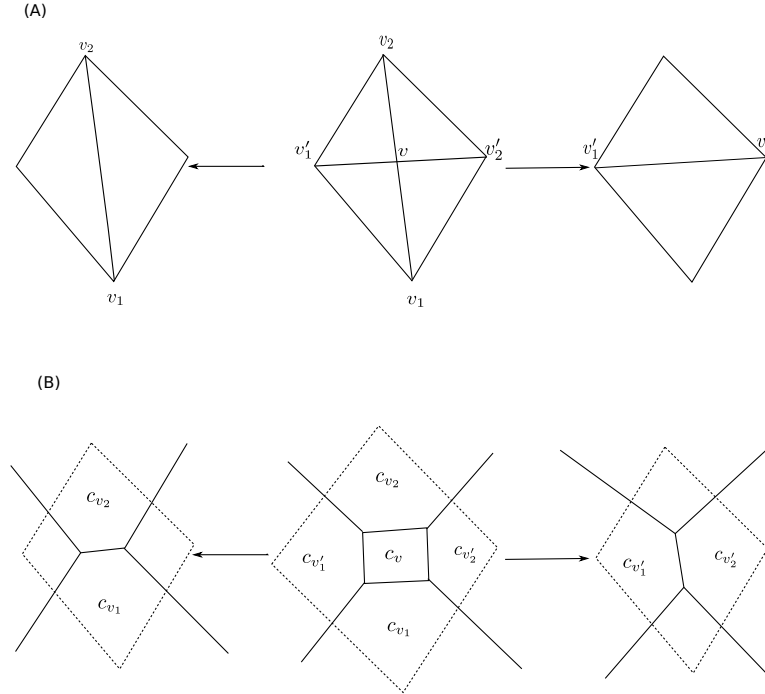
*Proof.* Consider a choice of cellular decomposition  $\mathcal{C}''$  dual to the common refinement  $\Delta''$ , in such a way that  $\mathcal{U}_u = \mathcal{U}_\sigma = \mathcal{U}_{\sigma'}$ , and  $\mathcal{C}''$  coincides with  $\mathcal{C}$  and  $\mathcal{C}'$  on  $M \setminus \mathcal{U}_v$ . Make  $\mathcal{C}''$  correspond with  $\mathcal{C}_{1/2}''$ . Using the fact that  $\overline{c_v} \cong \overline{c_\sigma} \times \overline{c_{\sigma'}}$ , define a family on  $(0, 1/2]$  by letting  $\overline{c_v}^t$  degenerate to  $\overline{c_\sigma}$ , and on  $[1/2, 1)$  by letting  $\overline{c_v}$  degenerate to  $\overline{c_{\sigma'}}$ , in such a way that all cells outside  $\mathcal{U}_v$  remain constant, all cells neighboring  $c_v^t$  transform into the corresponding

cells neighboring  $c_\sigma$ ,  $c_{\sigma'}$ , and the full family over  $(0, 1)$  is smooth (figure 4). Observe that, in particular, in the case when  $k = 0$  (resp.  $k = n$ ), one of the two degenerations would not be present: topologically, the cellular decompositions for  $t \in (0, 1]$  (resp.  $t \in [0, 1)$ ) would be equivalent.  $\square$

*Remark 15.* The triangle-dual cellular decomposition  $\mathcal{C}_\Delta$  that the  $n$ -sphere acquires from its realization as the boundary of an  $(n+1)$ -simplex is actually a triangulation of  $S^n$ , and hence self-dual (that is, isomorphic to its own dual). An important consequence is the following geometric realization of the dual Pachner moves in  $(S^n, \mathcal{C}_\Delta)$ . Choose a  $k$ -cell  $c_\sigma$  in  $\mathcal{C}_\Delta$ . Then, there is a  $(n - k)$ -cell  $c_{\sigma'}$  in  $\mathcal{C}_\Delta$ , complementary to the interior of the star of  $c_\sigma$ . Upon the choice of a smooth hemisphere  $H$  in  $S^n$  separating  $c_\sigma$  and  $c_{\sigma'}$ , we can identify each of the two components in  $S^n \setminus H$  with the open sets  $\mathcal{U}_\sigma$  and  $\mathcal{U}_{\sigma'}$ . In fact, more can be said about the topology of the cell degenerations: It is possible to foliate  $S^n \setminus \{\overline{c_\sigma}, \overline{c_{\sigma'}}\}$  as a collection of cell subcomplexes, parametrized by  $(0, 1)$ , and each isomorphic to  $\partial \overline{c_v}$ . Such foliation can be extended to a foliation of the closed disk  $\mathbb{D}^{n+1}$ , with the leave  $\mathcal{L}_t$  corresponding to  $\overline{c_v^t}$ . This is the case since there exists a diffeomorphism

$$(0, 1) \times \overline{c_v} \cong \overline{\mathbb{D}^{n+1}} \setminus \{\overline{c_\sigma}, \overline{c_{\sigma'}}\}.$$

FIGURE 4. (A) Pachner move on a triangulated surface, with common refinement shown (center). (B) The corresponding degenerations on the dual cellular decomposition.



As an immediate corollary of Pachner's theorem and lemma 6, we obtain the following result, which is potentially useful to find algorithmically simple sequences of transformations between triangle-dual cellular decompositions.

**Corollary 4.** *Any two triangle-dual cell decompositions of  $M$  are related by a sequence of deformations and contractions of cells preserving the triangle-dual property.*

*Remark 16.* In the same way that two cellular decompositions  $\mathcal{C}$  and  $\mathcal{C}'$  in a given manifold  $M$ , related by a dual Pachner move, can be thought of as degenerations of a 1-parameter family of cellular decompositions  $\{\mathcal{C}_t''\}$ , a pair of choices of path groupoids  $\mathcal{P}_{\mathfrak{F}}$  and  $\mathcal{P}_{\mathfrak{F}'}$ , generated by complete families  $\mathfrak{F}$  and  $\mathfrak{F}'$  adapted to  $\mathcal{C}$  and  $\mathcal{C}'$ , may be understood as degenerations of a 1-parameter family of path groupoids

$$\left\{ \mathcal{P}_{\mathfrak{F}_t''} \right\}_{t \in (0,1)}$$

resulting from a 1-parameter family of path families interpolating  $\mathfrak{F}$  and  $\mathfrak{F}'$ , and adapted to  $\{\mathcal{C}_t''\}_{t \in (0,1)}$ . The characteristic feature of such degenerations follows from the isomorphism of cell complexes  $\overline{c_v} = \overline{c_\sigma} \times \overline{c_{\sigma'}}$ . Namely, for any pair of subcells  $c_{\sigma_0} \subset \overline{c_\sigma}$  and  $c_{\sigma'_0} \subset \overline{c_{\sigma'}}$ , the subcell of  $\overline{c_v}$  whose closure corresponds to  $\overline{c_\sigma} \times \overline{c_{\sigma'}}$  would degenerate to  $c_{\sigma_0}$  or  $c_{\sigma'_0}$ . Such degeneration implies the *fusion* of the local path families supported over the closures of any two subcells in  $\overline{c_v}$  collapsing to the same subcell in  $\overline{c_\sigma}$  (or  $\overline{c_{\sigma'}}$ ).

Consider a pair of triangle-dual cell decompositions  $\mathcal{C}$  and  $\mathcal{C}'$ , related by a dual Pachner move as before, and moreover, assume they coincide over  $M \setminus \mathcal{V}_\sigma = M \setminus \mathcal{V}_{\sigma'}$ , where  $\mathcal{V}_\sigma$  (resp.  $\mathcal{V}_{\sigma'}$ ) denotes the open set in  $M$  given as the interior of the star of  $\overline{c_\sigma}$ —the union of all cells in  $\mathcal{C}$  (resp.  $\mathcal{C}'$ ) whose closures intersect  $\overline{c_\sigma}$  (resp.  $\overline{c_{\sigma'}}$ ). Let us moreover assume that  $\mathcal{C}$  and  $\mathcal{C}'$  are equipped with choices of extended lattice gauge fields  $\{\text{PT}_\mathcal{C}\}$  and  $\{\text{PT}_{\mathcal{C}'}\}$  coinciding over all common cells in  $M \setminus \mathcal{V}_\sigma = M \setminus \mathcal{V}_{\sigma'}$ . Consider, as before, a family  $\mathcal{C}_t''$  of triangle-dual cellular decompositions degenerating to  $\mathcal{C}$  and  $\mathcal{C}'$ , and such that for all  $t \in (0,1)$ , the open set  $\mathcal{V}_v$ —the interior of the star of  $\overline{c_v}$ —equals  $\mathcal{V}_\sigma = \mathcal{V}_{\sigma'}$ , and moreover, the restriction  $\mathcal{C}_t''|_{M \setminus \mathcal{V}_v}$  coincides with the restrictions  $\mathcal{C}|_{M \setminus \mathcal{V}_\sigma}$  and  $\mathcal{C}'|_{M \setminus \mathcal{V}_{\sigma'}}$ . Finally, equip such family with a smooth family of adapted path groupoids  $\{\mathcal{P}_{\mathfrak{F}_t''}\}$ , and degenerating to  $\mathcal{P}_{\mathfrak{F}}$  and  $\mathcal{P}_{\mathfrak{F}'}$ .

The next definition plays the role of a *generalized* cellular equivalence of extended lattice gauge fields, suited to consider smooth 1-parameter families of triangle-dual cellular decompositions arising from a dual Pachner move as before.

**Definition 13.** Two extended lattice gauge fields  $\{\text{PT}_\mathcal{C}\}$  and  $\{\text{PT}_{\mathcal{C}'}\}$  as before (that is, coinciding over all cells in  $M \setminus \mathcal{V}_\sigma = M \setminus \mathcal{V}_{\sigma'}$ ) are called *local*

relatives if there exists a smooth 1-parameter family

$$\left\{ \left\{ \text{PT}_{\mathcal{C}_t''} \right\} \right\}_{t \in (0,1)},$$

whose restriction to  $\mathcal{C}_t''|_{M \setminus \mathcal{V}_v}$  coincides with the respective restrictions of  $\{\text{PT}_{\mathcal{C}}\}$  and  $\{\text{PT}_{\mathcal{C}'}\}$  to  $\mathcal{C}|_{M \setminus \mathcal{V}_\sigma}$  and  $\mathcal{C}'|_{M \setminus \mathcal{V}_{\sigma'}}$ , and such that

$$\lim_{t \rightarrow 0^+} \left\{ \text{PT}_{\mathcal{C}_t''} \right\} = \{\text{PT}_{\mathcal{C}}\}, \quad \lim_{t \rightarrow 1^-} \left\{ \text{PT}_{\mathcal{C}_t''} \right\} = \{\text{PT}_{\mathcal{C}'}\}.$$

**Theorem 3.** *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two triangle-dual cellular decompositions of  $M$ , related by a dual Pachner move, together with a pair of extended lattice gauge fields  $\{\text{PT}_{\mathcal{C}}\}$  and  $\{\text{PT}_{\mathcal{C}'}\}$  such that the respective restrictions of their cores to  $\mathcal{C}|_{M \setminus \mathcal{V}_\sigma}$  and  $\mathcal{C}'|_{M \setminus \mathcal{V}_{\sigma'}}$  coincide. The principal  $G$ -bundles  $P$  and  $P'$  on  $M$ , induced by the cores  $\{\text{PT}_{\mathcal{C}}\}_{\min}$  and  $\{\text{PT}_{\mathcal{C}'}\}_{\min}$ , are equivalent if and only if  $\{\text{PT}_{\mathcal{C}}\}$  and  $\{\text{PT}_{\mathcal{C}'}\}$  are local relatives.*

*Proof.* The proof is straightforward, and its essence is the fact that, when the cores of the extended lattice gauge fields are considered, or what is the same, the relative homotopy classes of collections of clutching maps, the notion of local relativity for a pair of extended lattice gauge fields is nothing but a (local) specialization of the notion of cellular equivalence, adapted to degenerating families of cell decompositions. The extra complications and subtleties implicit in the notion of (global) cellular equivalence, present in general, disappear, since the homotopy equivalence relation that defines it has been confined to the  $(n-1)$ -cells in  $\overline{\tau_v}$  and the interior of its star.  $\square$

## 8. 't HOOFT LOOP OPERATORS

The 't Hooft loop operator was originally formulated in the seminal work [22], in the context of quantum gauge theory, as an operator that could sense non-local aspects of gauge fields. Its significance is due to the fact that it provides a so-called “disorder parameter”, and as such, it plays an essential role to study the phase diagram of quantum chromodynamics and in understanding the phenomenon of quark confinement. In this section we will study some of the topological and bundle theoretical aspects that are relevant in the mentioned construction (cf. [19]).

In its simplest form, the 't Hooft operation is defined on the space of smooth,  $G$ -valued holonomy maps over a space of thin homotopy classes of loops based at  $x_0 \in M$ , as in [3]. Let us consider an  $(n-2)$ -dimensional submanifold  $L \subset M$ <sup>12</sup> not containing  $x_0$ , and a smooth holonomy map  $\text{Hol}$  on  $(M, x_0)$ . The 't Hooft loop operation  $T_L$  is then constructed by modifying  $\text{Hol}$  into a new holonomy map  $\text{Hol}^L$  on  $M \setminus L$ , according to the recipe

$$(8.1) \quad \text{Hol}^L(l) = T_L[\text{Hol}](l) = g^{N(L,l)}[\text{Hol}](l),$$

<sup>12</sup> $L$  will be called an “ $(n-2)$ -loop”, or “ $(n-2)$ -knot”, or “Dirac  $(n-2)$ -string”, because in its original context, it lies inside a three-dimensional (“constant time”) slice of a four-dimensional Lorentzian manifold.

where  $g \in Z(G) \subset G$  is an element of the group's center that is arbitrary but fixed, and  $N(L, l)$  denotes the linking number<sup>13</sup> between  $L$  and  $l$  (the linking number of these two different types of objects is well-defined, and in fact, a topological invariant, under isotopies of  $l$  in  $M \setminus L$ ). The previous definition is, in fact, enough to determine the action of the operation in the space of smooth holonomy maps on  $M \setminus L$ .

The main goal of this section is to use our newly developed tools, to address the natural question of whether the principal  $G$ -bundles over  $M \setminus L$ , determined by  $\text{PT}_{\mathcal{C}}$  and  $\text{T}_L[\text{PT}_{\mathcal{C}}]$ , are equivalent. As we will see, such is the case, as a consequence of the connectedness of  $G$ , but in fact, while there is no longer a principal  $G$ -bundle structure over  $L \subset M$ , the defect of the cocycle condition over  $L$  turns out to be a manifestation of a more general topological structure, known as a *non-abelian bundle gerbe* [2, 4]. Thus, we are led to conclude that the natural setting to study the 't Hooft operation is not the category of principal  $G$ -bundles with connection, but instead a suitable category of non-abelian gerbes with the so-called “connective structures”.

Our starting point will be to extend equation (8.1) above, in order to define the 't Hooft operator in terms of smooth cellular parallel transport maps instead. Namely, let us consider a triangle-dual cell decomposition  $\mathcal{C}$  on  $M$ , a collection of cell base points, and the corresponding path groupoid  $\mathcal{P}_{\mathcal{C}}$ . Recall that if a cellular network  $\Gamma$  is chosen, then from every cellular parallel transport map  $\text{PT}_{\mathcal{C}}$ , we can extract the parallel transport data along the links contained in the discrete path subgroupoid  $\mathcal{P}_{\Gamma}^0$ , and that if a network parallel transport data is complemented with the homotopy cellular data, extracted from a minimal collection of path families  $\mathfrak{F}_{\min}$ , as in Section 4, we obtain sufficient information to characterize the bundle structure encoded in  $\text{PT}_{\mathcal{C}}$ .

In order to define the 't Hooft operation on the space of smooth cellular parallel transport maps, we will henceforth consider a pair  $(M, \mathcal{C})$ , and also consider exclusively the  $(n - 2)$ -loops  $L$  that fit in the  $(n - 2)$ -skeleton of  $\mathcal{C}$  (or conversely, if such a choice of  $(n - 2)$ -loop  $L \subset M$  is made, consider a triangle-dual cell decomposition  $\mathcal{C}$  with the previous property). As opposed to the previous definition of the 't Hooft loop operation, we will now restrict to *fixed* Seifert hypersurfaces  $S$  for  $L$ , satisfying the following property: consider a hypersurface  $S_0$  fitting in the  $(n - 1)$ -skeleton of  $\mathcal{C}$ , such that  $\partial S_0 = L$ , and let  $S$  be a sufficiently small isotopy deformation of

<sup>13</sup>The linking numbers  $N(L, l)$  can be calculated by means of the so-called *Seifert hypersurfaces*  $S$ , having  $L$  as boundary, as the intersection number of  $S$  and  $l$ , as the latter is independent of the choice of  $S$ , and which is defined due to the orientation in  $M$ . The existence of such hypersurfaces for any given 1-loop  $L$  is a classical result of Pontryagin-Frankl [6], and Seifert [21], in the cases  $M = S^3$  or  $M = \mathbb{R}^3$ , where in fact the linking number can be prescribed in terms of the celebrated Gauss linking integral. The definition of the linking number can be generalized analogously to dimensions higher than three [18]. Moreover, the linking number is invariant under thin homotopies, and descends to the group of based loops on  $(M, x_0)$ , modulo thin homotopy, and hence, it can be understood algebraically as a homomorphism from such group to  $\mathbb{Z}$ .

$S_0$ , preserving the boundary constraint, such that no base point for  $\mathcal{C}|_{M \setminus L}$  is contained in  $S$ , and such that there always exists a complete collection of path families  $\mathfrak{F}$  whose intersection numbers with  $S$  are well-defined (remark 17), and coincide with the intersection numbers of  $S_0$  for all the path families in  $\mathfrak{F}$  not involving cells in  $S_0$ . We will say that  $S$  is *neighboring*  $S_0$ .

Let us now consider a loop  $l$  that admits a factorization as a product of paths

$$l = \gamma_r \cdots \gamma_1.$$

with  $\gamma_1, \dots, \gamma_r \in \mathcal{P}_{\mathcal{C}}$ . Indeed, it is also possible to define intersection numbers  $N(S, \gamma_i)$ , provided that each  $\gamma_i$  intersects  $S$  transversally, in such a way that the usual linking number  $N(L, l)$  can be recovered from them. In particular, the following formula holds

$$N(L, l) = N(S, l) = \sum_{i=1}^r N(S, \gamma_i).$$

**Definition 14.** For  $L$  an  $(n-2)$ -dimensional submanifold of  $M$  embedded in  $\text{Sk}_{n-2}(\mathcal{C})$ ,  $S$  a fixed choice of Seifert hypersurface for  $L$  (i.e.  $\partial S = L$ ), neighboring some hypersurface  $S_0$  embedded in  $\text{Sk}_{n-1}(\mathcal{C})$ , and  $g \in Z(G)$  a fixed element of the center of  $G$ , the 't Hooft operation  $T_{L,S}$  on a smooth cellular parallel transport map  $\text{PT}_{\mathcal{C}}$  on  $M$  is defined as the cellular parallel transport map  $\text{PT}_{\mathcal{C}}^{L,S}$  on  $M \setminus L$  given by the formula

$$\text{PT}_{\mathcal{C}}^{L,S}(\gamma) = T_{L,S}[\text{PT}_{\mathcal{C}}](\gamma) = g^{N(S,\gamma)} [\text{PT}_{\mathcal{C}}](\gamma).$$

Observe that  $T_{L,S}$  is well defined in  $\mathcal{P}_{\mathcal{C}}|_{M \setminus L}$ , and when  $l$  is a loop in  $\mathcal{P}_{\mathcal{C}}$ , the parallel transports  $\text{PT}_{\mathcal{C}}^{L,S}(l)$  are independent of the choice of Seifert hypersurface  $S \subset M$ , and if, moreover,  $l = \gamma_r \cdots \gamma_1$ , then

$$\text{PT}^L(l) = \text{PT}^{L,S}(\gamma_r) \cdots \text{PT}^{L,S}(\gamma_1),$$

for any Seifert hypersurface  $S \subset M$ .

*Remark 17.* The 't Hooft operation is, in principle, only defined for the parallel transport of elements  $\mathcal{P}_{\mathcal{C}}$  that do not intersect  $L$ . However, if we now consider, for any pair  $c_v, c_w \in \mathcal{C}_n$  such that  $\overline{c_v} \cap \overline{c_w} \neq \emptyset$ , the cellular path families  $\mathcal{F}_{vw}$  defined in section 3, then, all paths  $\gamma_{vw}^x$  with  $x$  in the interior of  $\overline{c_v} \cap \overline{c_w}$  would have a well-defined linking number  $N(S, \gamma_{vw}^x)$ . In such sense, we can define the linking number for *all* elements in the family  $\mathcal{F}_{vw}$  (that is, *even* if  $x$  happens to lie in  $L$ ), as an integral invariant of the intersection cell  $\overline{c_{\tau}} = \overline{c_v} \cap \overline{c_w}$ , whose value is equal to 0, if  $\overline{c_{\tau}}$  is not contained in  $S$ , or 1, or  $-1$ , otherwise. We will denote such invariant as

$$\text{Or}(S, c_{vw}).$$

Observe that, in particular,  $\text{Or}(S, c_{vv}) = -\text{Or}(S, c_{vw})$ . The definition of the invariants  $\text{Or}(S, c_{vw})$  is crucial, as the clutching maps were precisely defined in terms of the path families  $\mathcal{F}_{vw}$ . In particular, given a cellular parallel



transport map  $\text{PT}_{\mathcal{C}}$ , with induced clutching maps  $\{h_{vw}(x) = \text{PT}_{\mathcal{C}}(\gamma_{vw}^x)\}$ , the 't Hooft operation induces the new collection of maps

$$\left\{ h_{vw}^{L,S} = g^{\text{Or}(S, c_{vw})} h_{vw} : \overline{c_\tau} \rightarrow G \right\},$$

which will be the main object of interest. As we will prove in theorem 4, their restriction to  $M \setminus L$  are clutching maps, thus determining a smooth principal bundle over  $M \setminus L$ . To understand the behavior of the collection  $\{h_{vw}^{L,S}\}$  over  $L$ , we require to recall the notion of non-abelian gerbe. Let  $(G, H, \delta, \alpha)$  be a cross module,<sup>14</sup> i.e.,  $G$  and  $H$  be groups, together with a pair of homomorphisms

$$\delta : H \rightarrow G, \quad \alpha : G \rightarrow \text{Aut}(H),$$

and such that for all  $g \in G$ ,  $h, h' \in H$ ,

$$\delta(\alpha(g)(h)) = g\delta(h)g^{-1}, \quad \alpha(\delta(h))(h') = h(h')h^{-1}.$$

Given an open cover  $\mathfrak{U} = \{\mathcal{U}_v\}$  of  $M$ , a *non-abelian bundle gerbe*<sup>15</sup> [2, 4] is a collection of smooth maps

$$g_{uv} : \mathcal{U}_{uv} \rightarrow G, \quad f_{uvw} : \mathcal{U}_{uvw} \rightarrow H,$$

satisfying

$$(8.2) \quad g_{uv}|_{\mathcal{U}_{uvw}} \cdot g_{vw}|_{\mathcal{U}_{uvw}} = \delta(f_{uvw}) \cdot g_{uv}|_{\mathcal{U}_{uvw}},$$

and

$$(8.3) \quad \alpha(g_{uv}|_{\mathcal{U}_{uvwt}})(f_{vwt}|_{\mathcal{U}_{uvwt}}) \cdot f_{uvt}|_{\mathcal{U}_{uvwt}} = f_{uvw}|_{\mathcal{U}_{uvwt}} \cdot f_{uwt}|_{\mathcal{U}_{uvwt}}.$$

**Theorem 4.** *The collection of maps*

$$(8.4) \quad \{h_{vw}^{L,S}\}_{\{c_v, c_w \in \mathcal{C}_n : \overline{c_v} \cap \overline{c_w} \neq \emptyset\}}$$

define a new collection of clutching maps for  $(M \setminus L, \mathcal{C}|_{M \setminus L})$ . The two collections of clutching maps are cellularly equivalent over  $M \setminus L$ ,

$$[\{h_{vw}^{L,S}\}]|_{M \setminus L} = [\{h_{vw}\}]|_{M \setminus L},$$

thus defining the same equivalence class of principal  $G$ -bundles on  $M \setminus L$ . Moreover, the collection (8.4) determines clutching maps for a non-abelian gerbe on  $M$ .

<sup>14</sup>In what follows, we will simply consider the case when  $H = G$  is a connected Lie group, or  $H = Z(G)$ , the center of  $G$ ,  $\delta$  is the identity or the inclusion map, and  $\alpha$  is the adjoint action.

<sup>15</sup>Originally, the notion of an (abelian) gerbe was introduced by Giraud in [7] (see [8] for a comprehensive and pedagogical introduction to the subject). The theory of gerbes has been vastly developed over the last decades, with applications spanning over a considerably broad variety of fields in Mathematics and Physics. Our approach to gerbes in this work is rather rudimentary, considering only what is strictly necessary for our purposes.



*Proof.* As before, let us fix a Seifert surface  $S_0$  for  $L$  contained in  $\text{Sk}_{n-1}(\mathcal{C})$ , together with a neighboring Seifert surface  $S$ . Recall that in a triangle-dual cellular decomposition  $\mathcal{C}$ , the closure of every  $(n-2)$ -cell  $c_\sigma$  is equal to the intersection of the closures of a triple of  $(n-1)$ -cells  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$ . Since a map  $h_{vw}^{L,S}$  differs from  $h_{vw}$  only when the  $(n-1)$ -cell closure  $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$  is contained in  $S_0$ , and an  $(n-2)$ -cell  $c_\sigma$  is contained in  $S_0$  if and only if at least one  $c_{\tau_i}$  (but at most two) is contained in  $S_0$ , the cocycle condition will automatically hold over all  $(n-2)$ -cells not contained in  $S_0$ , the new clutching maps are identical to the original collection over  $M \setminus S$ , and the induced bundles would coincide there.

Assume now that the  $(n-2)$ -cell  $c_\sigma$  is contained in  $S_0$ . An  $(n-2)$ -cell in  $S_0$  may either belong to  $S_0 \setminus L$  or to  $L$ . In the first case, exactly two of the  $(n-1)$ -cells  $\{c_{\tau_i}\}$  will be contained in  $S_0$ , while in the second case, exactly one of the  $(n-1)$ -cells  $\{c_{\tau_i}\}$  will be contained in  $S_0$ .

Let us first consider a closed  $(n-2)$ -cell  $\overline{c_\sigma}$  in  $S_0 \setminus L$ , corresponding to the intersection of  $S_0$  and an  $(n-1)$ -cell  $\overline{c_{\tau_1}}$  that is not in  $S_0$ . The remaining two  $(n-1)$ -cells  $\overline{c_{\tau_2}}, \overline{c_{\tau_3}}$  that intersect at  $\overline{c_\sigma}$  must belong to  $S_0$ , and their corresponding clutching maps would get modified. Due to the cyclic orientation that is given to  $(n-1)$ -cells incident in  $c_\sigma$ , the modification factor that is introduced by the 't Hooft operation in such pair of cells would appear with opposite exponents. Thus, the cocycle condition would be preserved in such case. Hence, by restriction, we obtain a collection of clutching maps over  $M \setminus L$ , which determine a new principal  $G$ -bundle there.

It remains to prove that the collection of clutching maps on  $M \setminus L$ , induced by  $T_{L,S}$ , is cellularly equivalent to the original collection. For this, we need to exhibit a homotopy between each pair of maps

$$\{h_{vw}, h_{vw}^{L,S}\},$$

in such a way that the compatibility condition is satisfied at every  $t \in [0, 1]$ . Consider any path  $g(t)$  in the group with  $g(0) = e$  and  $g(1) = g$ . The homotopy of pairs is defined in cases. When  $\text{Or}(S, c_{vw}) = 1$ , we define

$$h_{vw}^{L,S}(t)(x) = g(t)h_{vw}(x),$$

and when  $\text{Or}(S, c_{vw}) = -1$ , we define

$$h_{vw}^{L,S}(t)(x) = h_{vw}(x)g(t)^{-1}.$$

For the two of the three  $(n-1)$ -cells  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$ , whose closures intersect in  $\overline{c_\sigma}$  and belong to  $S_0$ , their associated clutching maps are modified. Now, according to the cyclic orientation, one of such maps is modified by a factor  $g(t)$  multiplying on the left, while the other one is modified by a factor  $g(t)^{-1}$  multiplying on the right. Thus, the modifications cancel, and the cocycle condition holds for every  $t \in (0, 1)$ . and the two collections of clutching maps are cellularly equivalent on  $M \setminus L$ .

Finally, let us assume that  $c_\sigma$  belongs to  $L$ . Then, only one of the corresponding three cells  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$  would belong to  $S_0$ , say  $c_{\tau_2}$ , and in such

case, the only map from the triple that would be modified would be  $h_{v_1 v_3}^{L,S}$ . Thus, the new compatibility condition for the triple reads

$$h_{v_1 v_2}^{L,S} \cdot h_{v_2 v_3}^{L,S} = g^{\text{Or}(S, c_{v_3 v_1})} h_{v_1 v_3}^{L,S},$$

hence, if we let

$$f_{v_1 v_2 v_3}^{L,S} = \begin{cases} g^{\text{Or}(S, c_{v_3 v_1})} & \text{if } c_\sigma \subset L, \\ e & \text{otherwise,} \end{cases}$$

we conclude that the new structure that we have obtained from the 't Hooft operation is a collection of clutching maps for a non-abelian gerbe, with a  $Z(G)$ -twisting supported over  $L$ : the remaining compatibility condition (8.3) follows from the fact that the homomorphism  $\alpha$  is trivial since  $g \in Z(G)$ , and moreover, an  $(n-3)$ -cell closure

$$\overline{c_u} \cap \overline{c_v} \cap \overline{c_w} \cap \overline{c_t}$$

has codimension 1 for the induced triangle-dual cell decomposition of  $L$ . Hence, the latter is the intersection of two  $(n-2)$ -cells in  $L$ , say  $\overline{c_{uvw}}, \overline{c_{uwt}}$ , and at the same time, the intersection of four  $(n-2)$ -cells  $\overline{c_{uvw}}, \overline{c_{uwt}}, \overline{c_{uwt}}, \overline{c_{vwt}}$  in  $M$ , so that  $c_{uwt}, c_{vwt} \not\subset L$ . Therefore, exactly 2 of the corresponding four maps  $\{f_{uvw}^{L,S}, f_{uwt}^{L,S}, f_{uwt}^{L,S}, f_{vwt}^{L,S}\}$  would be trivial, namely  $f_{uwt}^{L,S}, f_{vwt}^{L,S}$ , while the remaining two would provide canceling contributions in (8.3).

We could then talk about an induced isomorphism class of non-abelian gerbes

$$\{P^{L,S}\},$$

induced by the 't Hooft operation in  $\text{PT}_{\mathcal{C}}^{L,S}$ , and moreover,  $\text{PT}_{\mathcal{C}}^{L,S}$  should correspond to a gauge equivalence class of connective structures on  $\{P^{L,S}\}$  (but we will not dwell on the latter here).  $\square$

*Remark 18.* The previously described non-abelian gerbe structure admits an analogous description in terms of glueing maps. Also, observe that, since the 't Hooft operation is trivial over parallel transport maps with values on a connected Lie group with trivial center, it can be understood in terms of the fibration  $G \rightarrow G/Z(G)$  as a transformation on the space of non-abelian gerbes for  $G$  (containing in particular the space of Čech cocycles  $\check{Z}^1(M, \underline{G})$ ), fibering over the identity in  $\check{Z}^1(M, \underline{G/Z(G)})$ .

## APPENDIX A. CELLULAR DECOMPOSITIONS ON MANIFOLDS

Here we recall, for the sake of clarity and convenience, some standard notions on cell decompositions that we will require. We refer the reader to [11, 10] for further details.

Let  $M$  be an  $n$ -dimensional smooth manifold, assumed to be connected, not necessarily compact, and with or without boundary. A *smooth cellular*

*decomposition* of  $M$  is defined inductively, as a disjoint union of subsets

$$M = \bigsqcup_{c_\sigma \in \mathcal{C}} c_\sigma,$$

together with continuous maps  $\phi_\sigma^k : \overline{\mathbb{D}^k} \rightarrow M$  (where  $\mathbb{D}^k$  denotes the unit disk in  $\mathbb{R}^k$ ), such that each  $\phi_\sigma^k|_{\mathbb{D}^k}$  is a diffeomorphism onto  $c_\sigma$ , in such a way that  $c_\sigma$  is a  $k$ -dimensional cell, and moreover,  $\phi_\sigma^k(\partial\overline{\mathbb{D}^k})$  is a disjoint union of cells of dimension at most  $k-1$ . We will denote an arbitrary smooth cell decomposition of  $M$  by  $\mathcal{C} = \bigsqcup_{k=0}^n \mathcal{C}_k$ , where  $\mathcal{C}_k$  denotes the collection of  $k$ -dimensional cells in  $M$ . Similarly, the  $l$ th-skeleton of  $\mathcal{C}$  is defined as  $\text{Sk}_l(\mathcal{C}) = \bigsqcup_{k=0}^l \mathcal{C}_k$ .

A *flag* of length  $m+1 \leq n+1$  in a cell decomposition  $\mathcal{C}$  of  $M$  is a collection of nested cell closures  $\overline{c_{\sigma_0}} \subset \overline{c_{\sigma_1}} \subset \cdots \subset \overline{c_{\sigma_m}}$  in  $M$ . A flag is said to be *gapless* if  $\dim c_{\sigma_k} - \dim c_{\sigma_{k-1}} = 1$  for every  $1 \leq k \leq m$ . A flag is *complete* if its length is equal to  $n+1$ , so that  $\dim c_{\sigma_k} = k$ . A *descending flag* is defined in a similar way, by reversing the subset and cell dimension orderings.

In the case when  $M$  is orientable, and an orientation is chosen, there is a corresponding induced orientation in any cell  $c_\sigma \in \mathcal{C}$ , for any choice of a gapless flag starting at  $c_\sigma$ .

A barycentric subdivision  $B(\mathcal{C})$  of  $\mathcal{C}$  is any cellular decomposition of  $M$  satisfying the following property: there is a 1-1 correspondence between  $k$ -cells in  $B(\mathcal{C})$  and gapless flags of length  $k+1$  in  $\mathcal{C}$ , in such a way that if  $b_{\sigma_0 \dots \sigma_k}^k \in B(\mathcal{C})_k$  corresponds to  $\{\overline{c_{\sigma_0}} \subset \cdots \subset \overline{c_{\sigma_k}}\} \subset \mathcal{C}$ , then  $b_{\sigma_0 \dots \sigma_k}^k \subset c_{\sigma_k}$  (this is possible since  $\dim c_{\sigma_k} \geq k$  necessarily). Note how this definition is built inductively. In particular, there is exactly a 0-cell  $b_\sigma^0 \subset c_\sigma$  in  $B(\mathcal{C})$  for every  $c_\sigma \in \mathcal{C}$ .

A smooth triangulation of  $M$  is a homeomorphism  $\Delta : |K| \rightarrow M$ , where  $K$  is an abstract simplicial complex and  $|K|$  is its geometric realization, whose restriction to the interior of any simplex in  $|K|$  is a diffeomorphism. We will denote by  $\sigma, \tau, \dots$  the elements in  $K$  (called the *faces* of  $K$ ), and by  $v, w, \dots$  the vertices of it (in general, the maximal faces of an abstract simplicial complex  $K$  are called its *facets*, and  $K$  is said to be *pure* if all of its facets have the same dimension). Every triangulation has a canonically defined barycentric subdivision  $B(\Delta)$ .

*Remark 19.* The essential property that a cellular decomposition  $\mathcal{C}$  of  $M$  must satisfy to qualify as “sufficiently good” for our purposes (and to avoid unnecessary pathologies), is that for every  $k$ -cell  $c_\sigma$  ( $k \geq 1$ ),  $\partial\overline{c_\sigma}$  is a piecewise-smooth  $(k-1)$ -sphere (c.f. definition 10). Let us call such a broader class of cellular decompositions *spherical*. Clearly, every triangle-dual cellular decomposition is spherical (a property inherited from the P.L. structure), but the converse is not true: for instance, the hypercubical cellular decompositions of  $\mathbb{R}^n$ , determined by a lattice  $\Lambda \subset \mathbb{R}^n$ , are spherical but not triangle-dual, unless one consider the trivial case  $n = 1$ . It is thus

natural to inquire what special properties single out the triangle-dual cellular decompositions among the former. An intuitively obvious property is their genericity, that is, “almost all” spherical cellular decompositions are triangle-dual.<sup>16</sup>

**Lemma 7.** *Triangle-dual cellular decompositions are stable under small smooth deformations. Moreover, every spherical cellular decomposition is a degeneration of a smooth family of triangle dual cellular decompositions.*

*Proof.* A cellular decomposition  $\mathcal{C}_0$  would be unstable under small deformations if and only if there exists a family  $\mathcal{C}_\epsilon$ ,  $\epsilon \in \mathbb{R}$  of cellular decompositions which are combinatorially equivalent for small  $\epsilon \neq 0$ , and combinatorially inequivalent from  $\mathcal{C}_0$  (figure 5). If that is the case, then there exist at least one family of  $k$ -cells  $c_\sigma^\epsilon$ , for some  $0 < k < n$ , for which the limit of its closures  $\overline{c_\sigma^\epsilon}$  collapses to a 0-cell  $c_0$  in  $\mathcal{C}_0$ . In particular, all 0-cells in  $\overline{c_\sigma^\epsilon}$  coalesce into  $c_0$ . Over the dual cell decompositions  $\mathcal{C}_\epsilon^\vee$ , each 0-cell in  $\overline{c_\sigma^\epsilon}$  corresponds to an  $n$ -cell, and in the limit, all of these  $n$ -cells unify into the single  $n$ -cell  $c_0^\vee$ . It follows that the number of  $(n-1)$ -cells in  $\partial \overline{c_0^\vee}$  would be larger than the number of  $(n-1)$ -cells in the boundary of any of the  $n$ -cells in  $\mathcal{C}_\epsilon^\vee$  that are being unified into  $\partial \overline{c_0^\vee}$ .

If  $\mathcal{C}$  is triangle dual, then in particular  $\mathcal{C}^\vee$  would be a triangulation of  $M$ . By the previous argument,  $\mathcal{C}$  cannot be a degeneration of a smooth family of combinatorially equivalent cell decompositions  $\mathcal{C}_\epsilon$ , otherwise, there would be a family of  $n$ -cells in  $\mathcal{C}_\epsilon^\vee$  being unified into the interior of an  $n$ -simplex in  $\mathcal{C}^\vee$ , for which the number of  $(n-1)$ -cells in its boundary (equal to  $n+1$ ) is minimal, a contradiction.

Now, consider an arbitrary spherical cell decomposition  $\mathcal{C}$  of  $M$ . Passing to the dual, it is always possible to construct a refinement of  $\mathcal{C}^\vee$  which is triangulation  $\Delta$ , and whose vertices coincide with the set of 0-cells in  $\mathcal{C}^\vee$ , that is  $\text{Sk}_0(\mathcal{C}^\vee) = \text{Sk}_0(\Delta)$ . It is then clear that, since the  $n$ -cells in  $\Delta^\vee$  and  $\mathcal{C}$  are in 1-1 correspondence, it is possible to construct a family  $\mathcal{C}_\epsilon$  of combinatorially equivalent cell decompositions in  $M$ , such that for every sufficiently small  $\epsilon \neq 0$ ,  $\mathcal{C}_\epsilon$  is equivalent to  $\Delta^\vee$ , so by construction  $\mathcal{C}_\epsilon$  is triangle-dual, and such that the limit  $\mathcal{C}_0$  coincides with  $\mathcal{C}$ .  $\square$

## APPENDIX B. THE EQUIVALENCE THEOREM

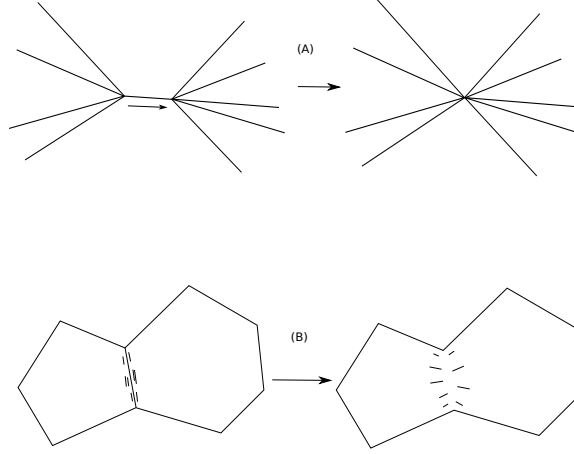
Recall that a principal bundle  $P$  can be determined over an open cover  $\mathfrak{U} = \{c_v\}_{v \in \mathcal{C}_n}$  by a Čech 1-cocycle, that is, a collection of smooth maps  $g_{vw} : \mathcal{U}_{vw} \rightarrow G$  satisfying the cocycle conditions

$$(B.1) \quad g_{vw} = g_{vw}^{-1} \quad \text{on } \mathcal{U}_{vw}, \quad g_{uv}g_{vw}g_{wu} = e \quad \text{on } \mathcal{U}_{uvw},$$

and any two such cocycles  $\{g_{vw}\}$  and  $\{g'_{vw}\}$  are equivalent if there exist a collection of smooth maps  $\{g_v : \mathcal{U}_v \rightarrow G\}_{c_v \in \mathcal{C}_n}$  (*local gauge transformations*)

<sup>16</sup>This statement and the subsequent lemma are meaningful when  $n \geq 2$ , since for 1-dimensional manifolds, all cellular decompositions are trivially triangle-dual.

FIGURE 5. (A) 1-cell contraction in a cellular decomposition  $\mathcal{C}$  on a surface. (B) The effect over the dual decomposition  $\mathcal{C}^\vee$ .



such that, over any  $\mathcal{U}_{vw}$ ,

$$(B.2) \quad g'_{vw} = g_v g_{vw} g_w^{-1}.$$

The cocycle condition is a statement on the 2-skeleton of the nerve  $N(\mathfrak{U})$ . In that sense, it will turn out to be very convenient to work with triangle-dual cellular decompositions and their associated open covers.

**Theorem 5.** *Let  $M$  be an oriented  $n$ -manifold,  $n \geq 2$ , together with a triangle-dual cellular decomposition  $\mathcal{C}$  and a compatible star-like cover  $\mathfrak{U}$  (see remark 1). There is a bijective correspondence*

$$\left\{ \begin{array}{l} \text{Cellular bundle data} \\ \text{in } M, \text{ relative to } \mathcal{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Equivalence classes of principal} \\ G\text{-bundles } P \rightarrow M \text{ trivialized over } \mathfrak{U} \end{array} \right\}$$

*Proof.* One implication is easy. Let us first consider an equivalence class of bundles with trivialization as a Čech 1-cocycle  $\{g_{vw}\}$  up to equivalence, as described above. For any  $c_\tau \in \mathcal{C}_{n-1}$ ,  $v, w \subset \tau$ , the restrictions

$$h_{vw} := g_{vw}|_{\overline{c_\tau}}$$

determine a representative of a class of cellular bundle data. This is so since every open set  $\mathcal{U}_v$  is contractible by definition, and hence any local gauge transformation  $g_v : \mathcal{U}_v \rightarrow G$  is smoothly homotopic to the identity. Therefore, any equivalent cocycle  $\{g'_{vw}\}$  would be homotopic to  $\{g_{vw}\}$  through a homotopy of cocycles, inducing a corresponding cellular equivalence of maps  $\{h_{vw}(t)\}$ ,  $t \in [0, 1]$ .

Conversely, let us consider an arbitrary choice of cellular bundle data  $\mathcal{D}$  over  $\mathcal{C}$ , and for each  $c_\tau \in \mathcal{C}_{n-1}$ , together with a choice of orientation, fix data representatives, in the form of maps  $h_{vw} : \overline{c_\tau} \rightarrow G$  such that for any cyclically oriented triples over any  $(n-2)$ -cell closure in  $\partial \overline{c_\tau}$  the restriction

to the diagonal  $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$  of the induced triples of maps lies in  $V_G$ . Our goal is to construct maps  $g_{vw} : \mathcal{U}_{vw} \rightarrow G$  such that for any cyclically oriented triple  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\} \in \mathcal{C}_{n-1}$  as before, and the corresponding open sets

$$\mathcal{U}_{\tau_1} = \mathcal{U}_{v_2} \cap \mathcal{U}_{v_3}, \quad \mathcal{U}_{\tau_2} = \mathcal{U}_{v_3} \cap \mathcal{U}_{v_1}, \quad \mathcal{U}_{\tau_3} = \mathcal{U}_{v_1} \cap \mathcal{U}_{v_2},$$

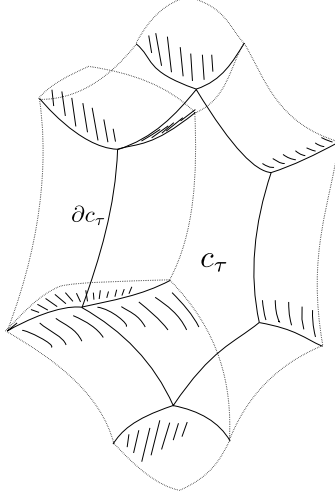
the cocycle conditions (B.1) are satisfied. The construction will be done in two steps.

For the first step, let us consider, for every  $c_\tau \in \mathcal{C}_{n-1}$ , the intersections  $\mathcal{U}_\tau \cap \text{Sk}_{n-1}(\mathcal{C})$  (figure 6), and their subsets

$$\mathcal{Z}_\tau = \mathcal{U}_\tau \cap \left( \bigcup_{\tau' \in \mathcal{C}_{n-1}^\tau} \overline{c_{\tau'}} \right),$$

where  $\mathcal{C}_{n-1}^\tau = \{c_{\tau'} \in \mathcal{C}_{n-1} : \overline{c_\tau} \cap \overline{c_{\tau'}} = \overline{c_\sigma}, c_\sigma \in \mathcal{C}_{n-2}\}$ . Each  $\mathcal{Z}_\tau$  is topologically a cylinder for  $\partial \overline{c_\tau}$  without boundary. We will extend each  $h_{vw}|_{\partial \overline{c_\tau}}$  to the whole  $\mathcal{Z}_\tau$  in such a way that the cocycle conditions are still satisfied. Consider any triple  $\{c_{v_1}, c_{v_2}, c_{v_3}\}$  as in (5.1). For each  $i = 1, 2, 3$

FIGURE 6. The intersection of  $\mathcal{U}_\tau$  with the  $(n-1)$ -skeleton of  $\mathcal{C}$ .



$j, k \neq i$ ,  $(ijk) \sim (123)$ , consider the intersection

$$\mathcal{J}_i = \text{pr}_i^{-1} (h_{v_j v_k}(\overline{c_{\tau_i}} \cap \mathcal{U}_\sigma)) \cap V.$$

Now, for  $i = 1, 2, 3$ , and  $c_\sigma$  as above, consider any collection of piecewise-smooth maps  $H_\sigma^i : (\overline{c_{\tau_1}} \cup \overline{c_{\tau_2}} \cup \overline{c_{\tau_3}}) \cap \mathcal{U}_\sigma \rightarrow G \times G \times G$ , satisfying that

- (i)  $H_\sigma^i((\overline{c_{\tau_1}} \cup \overline{c_{\tau_2}} \cup \overline{c_{\tau_3}}) \cap \mathcal{U}_\sigma) \subset \mathcal{J}_i$ ,
- (ii)  $H_\sigma^i|_{\overline{c_\sigma}} = (h_{v_1 v_2}, h_{v_2 v_3}, h_{v_3 v_1})|_{\overline{c_\sigma}}$ ,

(iii) If for  $j \neq i$  there is  $c_{\tau'_j}$  such that  $\text{int}(\overline{c_{\tau_j}} \cap \overline{c_{\tau'_j}}) \subset \mathcal{Z}_{\tau_i} \setminus \overline{c_{\tau_i}}$ ,<sup>17</sup> then

$$H_\sigma^i|_{\text{int}(\overline{c_{\tau_j}} \cap \overline{c_{\tau'_j}})} = H_{\sigma'}^i|_{\text{int}(\overline{c_{\tau_j}} \cap \overline{c_{\tau'_j}})}.$$

Such collections of maps would necessarily exist, since each one of the sets  $(\overline{c_{\tau_1}} \cup \overline{c_{\tau_2}} \cup \overline{c_{\tau_3}}) \cap \mathcal{U}_\sigma$  deformation retracts to  $\overline{c_\sigma}$ . The set of maps  $\{H_\sigma^i\}$  is parametrized by the elements  $c_\sigma \in \mathcal{C}_{n-2}$ , with a choice of cyclic orientation (5.1). Altogether, the functions  $\{H_\sigma^i\}_{c_\sigma \in \mathcal{C}_{n-2}}$  determine extensions of each  $h_{v_j v_k}$  to  $\mathcal{Z}_{\tau_i}$ ,  $(ijk) \sim (123)$ . The extension to a subset  $\mathcal{U}_{\tau_i} \cap \overline{c_{\tau_j}}$  is defined by means of the function  $H_\sigma^j$ . Conditions (ii) and (iii) above ensures that the overall process gives a well-defined extension of  $h_{v_j v_k}$  over  $\mathcal{Z}_{\tau_i}$ , and condition (i) ensures that the new functions would continue satisfying the cocycle condition (B.1). Let us continue denoting by  $h_{v_j v_k}$  such extension, now as a function over  $\mathcal{Z}_{\tau_i} \cup c_{\tau_i}$ .

The second step of the construction is to extend each  $h_{v_j v_k}$  to the whole  $\mathcal{U}_{\tau_i}$ , and relies in the following observation. For any  $c_\sigma \in \mathcal{C}_{n-2}$ , consider its triple  $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$ , which has been cyclically oriented. For  $i = 1, 2, 3$ , the sets

$$\mathcal{B}_{\tau_i} = \mathcal{U}_{\tau_i} \cap (c_{\tau_i} \cup c_{v_j} \cup c_{v_k}), \quad i \neq j, k, \quad j \neq k,$$

which are open cylinders for  $c_{\tau_i}$  (whose boundary points contain  $\mathcal{Z}_{\tau_i}$ ), satisfy  $\mathcal{B}_{\tau_1} \cap \mathcal{B}_{\tau_2} \cap \mathcal{B}_{\tau_3} = \emptyset$  (figure 6). The latter property implies that, if we consider an arbitrary piecewise-smooth extension of  $h_{v_j v_k}$  to  $\mathcal{B}_{\tau_i}$ , for every  $c_{\tau_i}$ , then, there is a unique way to further extend such maps to the sets  $\mathcal{U}_{\tau_i} \setminus \overline{\mathcal{B}_{\tau_i}}$  by forcing the cocycle condition to be satisfied on every triple intersection  $\mathcal{U}_\sigma$ , obtaining maps defined over  $\mathcal{U}_{\tau_i}$ . Hence, we can determine a collection of maps  $\{g_{vw} : \mathcal{U}_{vw} \rightarrow G\}$ , which define the cocycle we are looking for. We emphasize that the set of transition functions that we have constructed depends, a priori, on the choice of representatives  $h_{vw} : \overline{c_\tau} \rightarrow G$  for each  $c_\tau \in \mathcal{C}_{n-1}$  with a given orientation, and the choice of extensions of such  $h_{vw}$  to the sets  $\mathcal{Z}_\tau \cup \mathcal{B}_\tau$ .

To conclude the proof, we must show that, if we start with another set of representatives for our choice of cellular bundle data, or if we choose different extensions of the  $h_\tau$ , the new cocycle would be equivalent to the previous one. By the previous construction, any choice of extensions of any pair of representatives  $h_{vw}$  and  $h'_{vw}$  of a given class  $[h_{vw}]$  would be homotopic through cellularly-smooth functions satisfying the cocycle conditions, hence the resulting homotopies  $g_{vw}(t)$  between each  $g_{vw}$  and  $g'_{vw}$  would define transition functions for every  $t \in [0, 1]$ . Then, there is a well-defined principal  $G$ -bundle over  $M \times [0, 1]$ . It is a standard result, following the homotopy invariance properties of principal bundles [14], that the bundles resulting by restriction to the boundary are isomorphic.  $\square$

<sup>17</sup>Here,  $\text{int}(\overline{c_{\tau_j}} \cap \overline{c_{\tau'_j}})$  denotes the  $(k-2)$ -cell whose closure is  $\overline{c_{\tau_j}} \cap \overline{c_{\tau'_j}}$ .



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CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS A.C., JALISCO S/N, VALENCIANA,  
C.P. 36023, GUANAJUATO, GUANAJUATO, MÉXICO

*E-mail address:* `claudio.meneses@cimat.mx`

CENTRO DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO  
UNAM-CAMPUS MORELIA, A. POSTAL 61-3, MORELIA, MICHOACÁN, MÉXICO, AND  
DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO,  
ONTARIO, CANADA

*E-mail address:* `zapata@matmor.unam.mx`